MAU11602 Assignment 2, Due Wednesday 14 February 2024
Solutions

1. Consider a formal system with grammar

```
statement : number "+" number relation number
number    : "0" | number "'"
relation  : "=" | "≠"
```

the single axiom

```
0+0=0
```

and the following rules of inference

   (a) From any statement containing the strings `+` and `=0` we can deduce the same statement, but with `+` replaced by `'+` and `=0` replaced by `=0'`.
   (b) From any statement containing the string `=0` we can deduce the same statement, but with `=0` replaced by `'=0'`.
   (c) From any statement containing the strings `+` and `=0` we can deduce the same statement, but with `+` replaced by `'+` and `=0` replaced by `≠0`.
   (d) From any statement containing the string `=0` we can deduce the same statement, but with `=0` replaced by `'≠0`.
   (e) From any statement containing the string `=0` we can deduce the same statement, but with `=0` replaced by `≠0'`.

   Give formal proofs of the following theorems:

   (a) `0''+0'''=0'''''`
   (b) `0''+0''≠0'''''`

   Solution: Both proofs begin with

```
0+0=0
0'+0=0'
0''+0=0''
0''+0'=0'''
0''+0''=0''''
```

   The first one continues on with

```
0''+0'''=0'''''
```

   while the second one continues on with

```
0''+0''≠0'''''
```

2. Every string in the language from the previous problem is a `0` followed by some number of `'`s followed by a `+` followed by a `0` followed by some number of `'`s followed by either a `=` or a `≠` followed by a `0` followed by by some number of of `'`s. Say the numbers of `'`s above are j, k and l. A statement with a `=` is to be interpreted as meaning that $j + k = l$ while one with a `≠` is to be interpreted as meaning that $j + k \neq l$. Negation a statement means swapping a `=` for a `≠` or vice versa.

   With this interpretation, is the system

(a) sound?

(b) consistent?

(c) semantically complete?

(d) syntactically complete?

Note: You can't give formal proofs that any of these either are or aren't true since we don't have a formal system for describing formal systems. You can, and should, give informal proofs, but these don't have to be very detailed and you can use basic facts about arithmetic.
Solution: The system is sound. The axiom is true. The first two rules take a statement meaning $j + k = l$ and either increment j and l or increment k and l. Either of these operations will give a true statement if you started from one. The other three rules increment only one of j, k or l and change the equality to an inequality. This is also fine.

The system is consistent. A statement and its negation must be of the form $j + k = l$ and $j + k \neq l$ in some order and at most one of these is true. The system is sound, and as a consequence any theorem is true so at most one of them is a theorem.

The system is not semantically complete. An example of a true statement which is not a theorem is `0+0≠0''`. The following argument for why this is not a theorem is more detailed than I would expect you to give. Suppose `0+0≠0''` is a theorem. Then it has a proof and `0+0≠0''` must be its last line. It's not an axiom, so it must be derived by a rule of inference from an earlier statement. It can't have been derived by either of the first two, since those give statements with a `=` rather than `≠`. It can't be the third one, since that would give a statement with a `'` before the `+`. It can't be the fourth one since that would give a statement with a `'` before the `≠`. It must therefore be the fifth one. The only statement which we could apply the fifth to to get `0+0≠0''` is `0+0=0'`, so this must be an earlier statement in the proof. If we stopped the proof there we would have a proof of `0+0=0'`. But $0 + 0 = 1$ is not a true statement and we've already seen our system is sound so `0+0=0'` is not a theorem.

Note that `0+0≠0''` is far from the only true statement which is not a theorem and this is not the only proof that `0+0≠0''` is not a theorem. If you're curious, the true statements which are not theorems in this system are precisely those whose interpretation is $j + k \neq l$ with values of j, k and l such that $|j + k - l| > 1$.

The system is not syntactically complete. We just saw that `0+0≠0''` is not a theorem. It's negation, `0+0≠0''`, is also not a theorem, since $0 + 0 = 2$ is false and the system is sound.