MAU11602-1 Solutions and comments. DO NOT GIVE TO EXAMS OFFICE!

Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Faculty of Science, Technology, Engineering and Mathematics

School of Mathematics

JF Maths **Semester 2 2025-2026**
SF JH

MAU11602 Computation Theory and Logic

21 April 2026 **SC** **17.00–19.00**

Prof. John Stalker

Instructions to Candidates:

Calculators or mathematical tables are permitted, but unlikely to be helpful.

Instructions for Invigilators:

Credit will be given for the best 3 questions answered.

1. (20 points)

   (a) (2 points) The following steps are listed in alphabetical order, but in what order should they be performed?

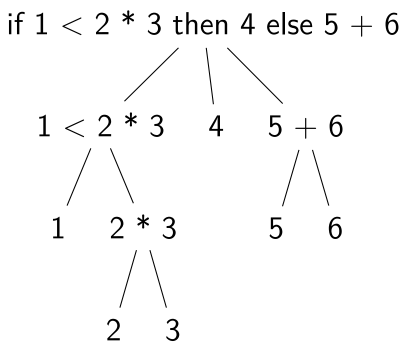
   - evaluation
   - lexing
   - parsing
   - type checking (or inference)

   Solution: Lexing, then parsing, then type checking (or inference), then evaluation.

   (b) (6 points) Find the parse tree for the expression

   ```
   if 1 < 2 * 3 then 4 else 5 + 6
   ```

   Solution:



   Comments: There were a few different grammars for infix arithmetic so you might get a slightly different parse tree. Any of them would be acceptable.

   (c) (6 points) Give a typing derivation to show that the expression

   ```
   if 1 < 2 * 3 then 4 else 5 + 6
   ```

   has type `int`.

   Solution:

$$\dfrac{\dfrac{1 : \mathsf{int} \qquad \dfrac{2 : \mathsf{int} \qquad 3 : \mathsf{int}}{2 * 3 : \mathsf{int}}}{1 < 2 * 3 : \mathsf{bool}} \qquad 4 : \mathsf{int} \qquad \dfrac{5 : \mathsf{int} \qquad 6 : \mathsf{int}}{5 + 6 : \mathsf{int}}}{\mathsf{if}\ 1 < 2 * 3\ \mathsf{then}\ 4\ \mathsf{else}\ 5 + 6 : \mathsf{int}}$$

   (d) (6 points) Show all reduction steps in evaluating the expression

```
if 1 < 2 * 3 then 4 else 5 + 6
```

Solution:

```
if 1 < 2 * 3 then 4 else 5 + 6

if 1 < 6 then 4 else 5 + 6

if true then 4 else 5 + 6

4
```

2. (20 points)

   (a) (8 points) Consider the following expression

$$\forall x.\forall \epsilon.\epsilon > 0 \to (\exists \delta.\delta > 0 \wedge (\forall y.\,(|x-y| < \delta) \to (|f(x) - f(y)| < \epsilon)))\,.$$

   For each of the following subexpressions, which occurences of which variables are free and which are bound?

   i. the whole expression,

   ii. the subexpression

$$\epsilon > 0 \to (\exists \delta.\delta > 0 \wedge (\forall y.\,(|x-y| < \delta) \to (|f(x) - f(y)| < \epsilon)))\,,$$

   iii. the subexpression

$$\delta > 0 \wedge (\forall y.\,(|x-y| < \delta) \to (|f(x) - f(y)| < \epsilon))\,,$$

   iv. the subexpression $|f(x) - f(y)| < \epsilon$.

   Solution:

   i. All occurences of f are free and all occurences of x, y, δ and ϵ are bound.

   ii. All occurences of f, x and ϵ are free and all occurences of y and δ are bound.

   iii. All occurences of f, x, δ and ϵ are free and all occurences of y are bound.

   iv. All occurences of f, x, y and ϵ are free.

   Comments: Technically f is not a variable, so if you omit it from all of these that's also fine.

   (b) (4 points) Give an example of an expression and a variable which has both free and bound occurences of that variable. If this happened in the previous part you can reuse that expression. Otherwise you have to find a new example.

   Solution:

$$x = 0 \wedge \exists x.x > 0$$

   Comments: There are many possible answers but the one above is one of the shortest.

(c) (8 points) Perform a capture avoiding substitution of j for i and $2j$ for k in

$$\sum_{i=j}^{k} i + \sum_{j=i}^{k} j = \frac{(j+k+1)(k-j)}{2} + \frac{(i+k+1)(k-i)}{2}$$

Solution: We need to perform α conversion to avoid variable capture. It doesn't matter what which fresh variables we substitute for i in the first sum and for j in the second but I'll choose l and m:

$$\sum_{l=j}^{k} l + \sum_{m=i}^{k} m = \frac{(j+k+1)(k-j)}{2} + \frac{(i+k+1)(k-i)}{2}.$$

Now we can safely substitute:

$$\sum_{l=j}^{2j} l + \sum_{m=j}^{2j} m = \frac{(j+2j+1)(2j-j)}{2} + \frac{(j+2j+1)(2j-j)}{2}.$$

Comments: You can obviously simplify this further, but you weren't asked to.

3. (20 points)

   (a) (4 points) What is an arithmetic function?

   Solution: A function f from the natural numbers is said to be arithmetic if there is an expression of boolean type with free variables m and n which is satisfied precisely by those m and n for which $n = f(m)$.

   (b) (4 points) Give an example of a function from the natural numbers to the natural numbers which is not arithmetic. You don't need to prove that your example works, as long as I can see that it does.

   Solution: By Tarski's theorem the set of encodings of true statements in Peano arithmetic is not an arithmetic set. We need a non-arithmetic function rather than a non-arithmetic function though, so we have to do a bit more work. We can, for example, define a function f such that $f(m) = 1$ if m encodes a true statement in Peano arithmetic and $f(m) = 0$ otherwise. Suppose f were arithmetic and let P be the statement in Peano arithmetic expressing $n = f(m)$. Then $P \wedge n = S0$ has m as a free variable and expresses the fact that m is the encoding of a true statement, contradicting Tarski's theorem.

   Comments: There are other ways to use Tarski's theorem, but you won't be able to avoid using it in some way.

   (c) (4 points) Prove that there is a function from the natural numbers to the natural numbers which is not arithmetic. You can use your answer to the previous part, but only if you actually proved that the function you gave there is not arithmetic, using only results proved in lecture.

   Solution: Tarski's theorem was stated in lecture, but not proved in lecture, so it's not available. In other words, you can't use your answer from the previous part unless you prove Tarski's theorem, which I don't recommend, since it's much easier to use a counting argument.

   There is a relation between natural numbers and arithmetic functions expression the fact that the natural number encodes a Boolean expression with two free variables which holds if and only if the second is the value of the function evaluated at the first. There is at most one such function for each natural number and there is such a natural number for each arithmetic function so the relation is right unique and right total. The set of natural numbers is countable so the set of arithmetic functions must be total. The set of all functions is uncountable though. We can see this as follows. There is an obvious bijection between the subsets of the natural numbers and the functions which take only the values 0 and 1. The set of subsets is uncountable by Cantor's theorem and so the set of functions taking only the values 0 and 1 is uncountable. Subsets of countable sets are countable so the set of all functions from the natural numbers to the natural numbers must be uncountable.

Comments: This is a counting argument similar to several we've seen before, like the one for the existence of transcendental numbers. There are various other ways to state this, for example with a surjective function from a subset if you don't like to describe things in terms of relations.

(d) (4 points) Gödel's Incompleteness Theorem resulted from an ultimately unsuccessful attempt to prove what other theorem? Is that theorem now known to be true? Is it now known to be false?

Solution: Gödel was attempting to prove the inconsistency of Peano arithmetic. It is not known whether Peano arithmetic is inconsistent.

(e) (4 points) What is Rosser's Theorem, and how is it related to Gödel's Theorem?

Solution: Rosser's theorem says that Peano arithmetic cannot be consistent and syntactically complete. Gödels theorem says it can't be ω consistent and syntatically complete. Rosser's theorem implies Gödel's theorem.

4. (20 points)

   (a) (4 points) Let A^c denote the complement of the set A so that $x \in A^c$ is true if and only if $x \in A$ is false. I am just establishing notation here, not claiming that any such set exists. For which sets A does A^c exist?

   Solution: For no sets. This was proved in lecture.

   (b) (6 points) Justify your answer to the previous part.

   Solution: Suppose A^c existed for some A and use Union, Pairing and Separation to form the set $B = \{S \in A \cup A^c \colon S = \varnothing \vee \exists x. x \in S\}$. Then $S \in B$ if and only if S is a set. We can now proceed with the usual Russell paradox construction. Let $C = \{S \in B \colon S \notin S\}$. This exists by Separation. Then $C \in C$ implies $C \notin C$ and vice versa, so we have a contradiction.

   (c) (4 points) Regardless of what your answers were for the previous parts, when we establish a notation for an object whose existence we're at least temporarily unsure of we need to be careful which version of first order logic we use. Which rules of inference need to be modified to deal with this situation?

   Solution: The rules which require modification are the introduction rule for $\exists$ and the elimination rule for $\forall$.

   (d) (6 points) State and prove Cantor's Theorem.

   Solution: The theorem says that if A is a set then there is no surjective function from A to its power set $\mathcal{P}A$.

   To prove the theorem, suppose f is such a function and define

$$B = \{x \in A \colon \exists C \in \mathcal{P}A. C = f(x) \wedge x \notin C\}$$

   Now $B \in \mathcal{P}A$ and f is surjective so there's a $y \in A$ such that $B = f(y)$.

   If $y \in B$ then by the definition of B there's a $C \in \mathcal{P}A$ with $C = f(y)$ and $y \notin C$, so $B = f(y) = C$ and $y \notin B$.

   On the other hand, if $y \notin B$ then $C = B$ satisfies the conditions $C = f(y)$ and $y \notin C$, so y satisfies the membership conditions for B and so $y \in B$.

In other words y is a member of B if and only if it isn't. From this contradiction we see that there can be no such f.

Comments: This is, of course, the proof given in lecture.