# MAU11602 Lecture 25

John Stalker

Trinity College Dublin

# Foundation

The Axiom of Foundation says that every non-empty set has a member which is disjoint from it.

In symbols, $\forall A.[\exists B.B \in A] \supset [\exists B.[[B \in A] \wedge [A \cap B = \emptyset]]$.

This is also called the Axiom of Regularity.

There appears to be no intuition behind it.

Nothing particularly bad happens if you reject it.

It is nonetheless usually included in the Zermelo-Fraenkel axiom set, which is the most popular axiomatic system for set theory.

# Extensionality, revisited

The version of Extensionality we've used is "If $A$ and $B$ are sets and every member of $A$ is a member of $B$ and vice versa then $A = B$."

There are various equivalent formulations, e.g. we can replace "sets" with "non-empty sets" or assume only that $A$ or $B$ is a set.

Suppose we drop the condition that $A$ or $B$ is a set entirely.

Then everything is a set! The things you think aren't sets, e.g. my cat Clawdette, have no members and so are equal to the empty set!

Similarly, you are equal to the empty set, as am I!

The set of cats is $\{\emptyset\}$, as is the set of Trinity maths lecturers, so those two sets are the same!

# More extensionality

Bizarrely, mathematics would be largely unaffected. We've defined natural numbers as sets. We can (and will) define integers as sets of natural numbers, rational numbers as sets of integers, etc.

So if you only ever want to apply set theory in mathematics you *can* assume a version of Extensionality without the assumption that $A$ and $B$ are sets.

In fact it makes a few things slightly easier so mathematicians traditionally do this.

## ZF, ZFC, alternatives

As with first order logic there is really more than one set theory.

The most common is ZFC: the new (improved?) version of Extensionality, Elementary Sets, Separation, Power Set, Union, Replacement, Infinity, Choice and Foundation.

The next most common is ZF: all of the above minus Choice.

A minimalist version would be the original Extensionality, Elementary Sets, Separation, Power Set, Union, and Infinity.

A reasonable compromise is those plus Replacement and Dependent Choice.


Figure 1: A graph

## What is a graph?

Nothing to do with graphs of functions.

We have a set of “vertices” and a set of “edges”, each connecting a vertex to a vertex.

Questions:

- Can there be infinitely many vertices?
- Can there be infinitely many edges?
- Do the edges have a direction?
- Can there be an edge from a vertex to itself?
- Can there be more than one edge from one vertex to another?

There are really several different graph theories, depending on the answers to these questions.

Here we’ll assume yes, yes, yes, yes and no, but most of our graphs will be finite and we’ll represent undirected graphs as graphs where the edges come in pairs, with opposite directions.

We're mostly interested in graphs where the "vertices" represent states and the "edges" represent allowed state transitions.

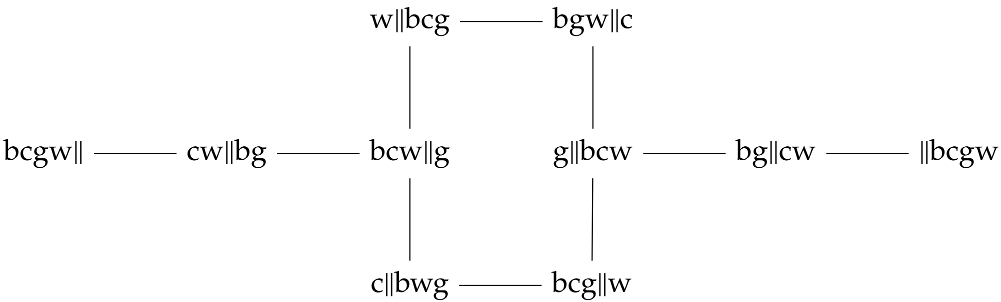



Figure 2: Transporting cabbages, goats and wolves

You could do the same thing for the Affenspiel puzzle, but it wouldn't fit on a slide.

Often the allowed transitions depend on some input, and the edges need to be labelled with the inputs which allow the transition.

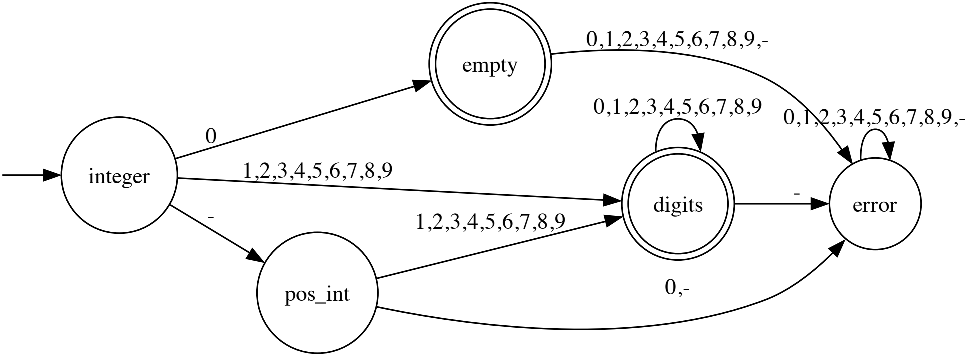
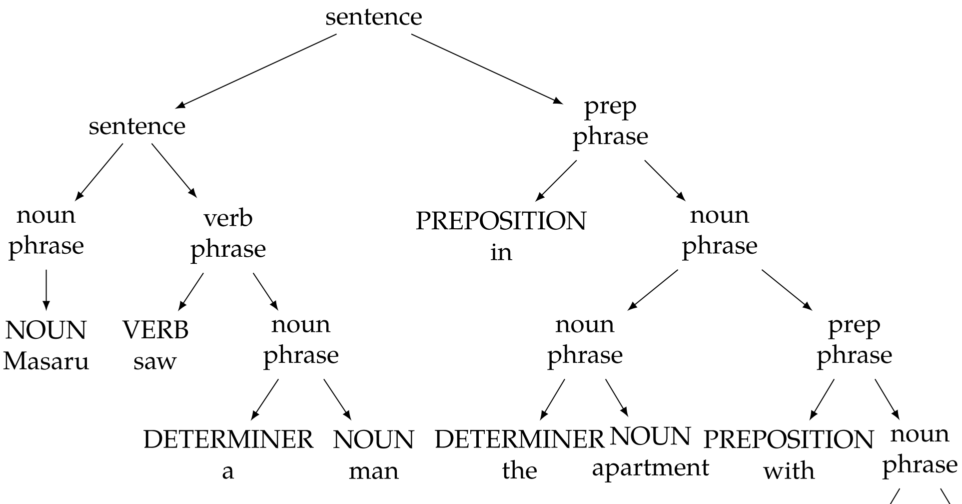



Figure 3: A strongly deterministic finite state automaton for the integers

## Trees are graphs

There are two types of trees, directed and undirected, just like there are directed and undirected graphs. The trees we've met are directed. Edges go away from the root towards the leaves.



# Graphs and choice

Do you believe the following statement?

If there is at least one edge from each vertex of a directed graph then for each choice of initial vertex there is at least one infinite walk in the graph starting at that vertex.

This statement is equivalent to the Axiom of Dependent Choice.

It's important here that we've allowed infinite graphs.

## Classical graph theory

Most of graph theory deals with finite undirected graphs with no self-loops.

In other words, graphs like the tube map or the cabbage/goat/wolf graph.

Questions:

- Connectedness
- Colouring
- Existing of Eulerian trails and circuits
- Existence of Hamiltonian paths and circuits

# Example theorem

For finite undirected graphs with no self-loops define the degree of a vertex to be the number of edges going in or out.

An Eulerian circuit is a path which traverses each edge exactly once and ends up at the same vertex it started at.

An old theorem, due to Euler, says that such a graph has an Eulerian circuit if and only if it's connected and all vertices have even degree.

As with most of graph theory, there is an algorithm for this, not just an existence proof.