MAU11602 Lecture 24

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Sizes of sets

We say A is no larger than B if there is an injection from A to B.

We say that A is of the same size as B if A is no larger than B and vice versa.

We say that A is strictly smaller than B if A is no larger than B but they are not of the same size.

Proper subsets of a finite set are strictly smaller than it, but this can fail for infinite sets. The set of even natural numbers is of the same size as the set of natural numbers, for example.

If there is a bijection from A to B then A and B are of the same size.

The converse is the Schröder-Bernstein Theorem. It's not obvious, although the proof is elementary.

Partial orders and equivalence relations

Is "is no larger than" a partial order?

- Reflexivity: A is no larger than A?
- Transitivity: If A is no larger than B and B is no larger than C then A is no larger than C?
- Antisymmetry: If A is no larger than B and vice versa then $A = B$?

The first two hold, but not the third. It would hold if we replaced $A = B$ with A is of the same size as B.

Technically "is no larger than" isn't even a relation, since there is no set of all sets.

"Is of the same size" as is reflexive, transitive and symmetric, but not technically an equivalence relation, for the same reason.

Power sets

Every set is strictly smaller than its power set.

A is no larger than PA since $f(x) = \{x\}$ is an injection from A to PA.

If PA were of the same size as A there would be a bijection from PA to A.

Call it g and define
$$C = \{x \in A : \exists D \in PA : g(x) \notin D\}.$$

g is surjective and $C \in PA$ so $C = g(y)$ for some $y \in A$. Is $y \in C$?

If it is then it isn't and vice versa.

So there is no such g. In other words PA is not of the same size as A and so A is strictly smaller than PA.

This argument is called the Cantor diagonal argument. It's similar to our argument that there is no set of all sets.

Levels of infinity

Let N be the set of natural numbers. Then N is strictly smaller than PN, which is strictly smaller than PPN, etc.

Sets which are not larger than N are called countable and sets which are not countable are called uncountable.

This terminology is not universal! Some people reserve "countable" for infinite sets.

Properties of countable sets:

- Finite sets are countable (with the convention above).
- Subsets of countable sets are countable (with the convention above).
- Finite unions, intersections and relative complements of countable sets are countable (with the convention above).
- Cartesian products of countable sets are countable.
- If A is countable then the set LA of lists whose elements are members of A is also countable.
- If A is countably infinite then PA is not countable.

Counting arguments

Suppose we have a set and a property (Boolean expression) which members of the set may or may not have, and

- the set is uncountable, but
- only countably many members have the property.

Then some member does not have the property.

Arguments of this form are called counting arguments. They appear often in mathematics and you need to learn how to read and write them.

Example of a counting argument

There is a subset of N which is not arithmetic.

Tarski's Theorem tells us that the set of encodings of true statements is not arithmetic, but a counting argument is much easier.

N is countably infinite so PN is uncountable.

Each arithmetic set has a Boolean expression which defines it. There might be (will be) more than one so choose the one with the smallest encoding.

This encoding gives an injection from the set of arithmetic sets to N, so there are countably many arithmetic sets.

There must therefore be a set which is not arithmetic.

Another example of a counting argument

There is a language which cannot be described by a grammar.

Recall that a language is a subset of LT, where T is the set of tokens, which we'll always take to be countable.

If T is non-empty then LT is countable and PLT, the set of languages with tokens in T, is uncountable.

A grammar for such a language is finitely describable, so there are at most countably many of them.

There must therefore be a language without a grammar.

Right inverses for surjective functions

Do you believe the following statement?

If f is a surjective function from a set A to a set B then there is an injective function g from B to A such that $f \circ g$ is the identity function on B.

Surjective means that for each $y \in B$ there is an $x \in A$ such that $f(x) = y$.

For each $y \in B$ choose such an x and call it $g(y)$.

Do you believe the following statement?

There are sets A_1, A_2, A_3, A_4, A_5, B_1, B_2, B_3, C_1, C_2, C_3, C_4, and C_5 in three dimensional Euclidean space such that

- The sets B_1, B_2 and B_3 are balls of equal radius.
- For each j the sets A_j and C_j are congruent.
- The A's are disjoint and their union is $B_1 \cup B_2$.
- The C's are disjoint and their union is B_3.

We expect that $\mathrm{Vol}(A_j) = \mathrm{Vol}(C_j)$, $\sum_{j=1}^{5} \mathrm{Vol}(A_j) = \mathrm{Vol}(B_1) + \mathrm{Vol}(B_2) = \frac{8}{3}\pi r^3$ and $\sum_{j=1}^{5} \mathrm{Vol}(C_j) = \mathrm{Vol}(B_3) = \frac{4}{3}\pi r^3$.

It follows that $1 = 2$.

Paradox, and ways to avoid it

The first statement I asked about is "obviously" true and the second is "obviously" false, but unfortunately the first one implies the second.

What are we to do?

We could abandon the idea that bounded subsets of Euclidean space have well-defined volumes, or we could abandon the idea that all surjective functions have right inverses.

Both are viable options.

In measure theory we identify a class of sets as measurable, and only define measure for those sets.

We could also formulate set theory without an Axiom of Choice, i.e. one which legitimises my "for each $y \in B$ choose such an x and call it $g(y)$" argument. It doesn't follow from our other axioms.

Set theory without an axiom of choice

The problematic statement was

If f is a surjective function from a set A to a set B then there is an injective function g from B to A such that $f \circ g$ is the identity function on B.

With the axioms we have we can still prove this if A is countable or B is finite.

That's not a problem, i.e. doesn't lead to the Banach-Tarski paradox.

We could even safely add an Axiom of Countable Choice, i.e. the case where B is countable.

We can go a bit further still, with the Axiom of Dependent Choice, which is weaker than the Axiom of Choice but stronger than the Axiom of Countable Choice.

Dependent Choice is enough for almost all of classical mathematics, and doesn't lead to the Banach-Tarski paradox.