MAU11602 Lecture 23

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Notes

I've posted the next two chapters of the notes, on Graph Theory and Abstract Algbra.

I won't cover those in detail and it's fine if you just skim them.

I've included them only because they shed some light on other topics and they're not examinable.

Where we left off

Last time I defined lists.

Once we have lists we can define Cartesian products. $A \times B$ is the set of (x, y) where $x \in A$ and $y \in B$.

It's not completely obvious this is a set, but it is.

Cartesian products of finite sets

If A and B are finite then so is $A \times B$.

This is proved by a double induction.

Suppose $C \subseteq A$ and define $F = \{D \in PB : [C \times B] \cup [\{x\} \times D] \text{ is finite}\}$.

If $C \times B$ is finite then $\emptyset \in F$.

If $D \in F$ then

$$[C \times B] \cup [\{x\} \times [D \cup \{y\}]] = [C \times B] \cup [\{x\} \times D] \cup \{(x, y)\}$$

is finite.

So $B \in F$ by induction.

In other words,

$$[C \cup \{x\}] \times B = [C \times B] \cup [\{x\} \times B]$$

is finite.

Proof, continued

We just saw that if $C \times B$ is finite and $x \in A$ then $[C \cup \{x\}] \times B$ is finite.

Define

$$E = \{C \in PA : C \times B \text{ is finite}\}.$$

So if $C \in E$ and $x \in A$ then $C \cup \{x\} \in E$.

Now $\emptyset \in E$ so, by induction, $A \in E$.

In other words, $A \times B$ is finite.

Relations

A relation is a set of lists, all of the same length.

Length 0 gives a nullary relation, not very interesting.

Length 1 gives a unary relation. They're essentially just sets.

Length 2 gives a binary relation. This is the most interesting case.

Length 3 gives a ternary relation. They come up in some places.

Length greater than 3 has few applications.

If $R \subseteq A \times B$ we say R is a relation *from* A *to* B. If $R \subseteq A \times A$ we say that R is a relation *on* A.

Important types of binary relation:

- Partial orders
- Equivalence relations
- Functions

General binary relations

Operations on binary relations:

- Inverse: $(x, y) \in R^{-1}$ if and only if $(y, x) \in R$.
- Composition: $(x, z) \in R \circ S$ if and only if there is a y such that $(x, y) \in S$ and $(y, z) \in R$.

These have the expected properties, e.g. $\left(R^{-1}\right)^{-1} = R$ and $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$.

Properties a binary relation R on A *could* have:

- Reflexive: $I_A \subseteq R$, where $I_A = \{(x, y) \in A^2 : x = y\}$. In other words, for all x in A we have $(x, x) \in R$.
- Symmetric: $R^{-1} = R$. In other words, if $(x, y) \in R$ then $(y, x) \in R$.
- Antisymmetric: $R \cap R^{-1} \subseteq I_A$. In other words, if $(x, y) \in R$ and $(y, x) \in R$ then $x = y$.
- Transitive: $R \circ R \subseteq R$. In other words, if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

Note that symmetric and antisymmetric are not opposites.

Partial orders and equivalence relations

An equivalence relation is a binary relation which is reflexive, symmetric and transitive.

A partial order is a binary relation which is reflexive, antisymmetric and transitive.

Equality, i.e. the relation I_A is an equivalence relation and a partial order.

The (non-strict) inequality relation $\leq$, i.e. $R = \{(x,y) \in N^2 : x \leq y\}$, is a partial order on the natural numbers, but not an equivalence relation (not symmetric).

The strict inequality relation $<$, i.e. $R = \{(x,y) \in N^2 : x < y\}$, is not a partial order (not reflexive).

Equality modulo p, i.e. $\{(x,y) \in N^2 : \exists z \in N : x = y + p \cdot z \vee y = x + p \cdot z\}$, is an equivalence relation on the natural numbers, but not a partial order (not antisymmetric).

More properties for binary relations

A binary relation R from A to B is called

- left unique if $R \circ R^{-1} \subseteq I_B$, i.e. if $(x, y) \in R$ and $(z, y) \in R$ imply $x = z$.
- right unique if $R^{-1} \circ R \subseteq I_A$, i.e. if $(x, y) \in R$ and $(x, z) \in R$ imply $y = z$.

The domain of R is the set $\{x \in A : \exists y \in B : (x, y) \in R\}$. The range of R is the set $\{y \in B : \exists x \in A : (x, y) \in R\}$.

R is called

- left total if its domain is A
- right total if its domain is B.

- a function if it is left total and right unique
- an injective function if it is a left unique function
- a surjective function if it is a right total function
- a bijective function if it is injective and surjective

Functions of more arguments

The functions considered above are functions of one argument, i.e. unary functions.

Nullary functions (functions of no arguments) are given by unary relations.

Binary functions (functions of two arguments) are given by ternary relations, etc.

Functions on finite sets

If R is an injective function from a finite set A to itself then R is also surjective, hence bijective.

This is proved by set induction. See the notes for details.

If we drop the finiteness assumption this is no longer true. The increment function on the natural numbers is a counter-example.

Axiom of Replacement

Suppose we have

- a Boolean expression P with free variables x and y.
- a set A

We'd like there to be a relation R from A to some set B such that $(x,y) \in R$ if and only if $x \in A$ and P holds and there is only one such value of y for each x.

If A is finite we can prove this by induction.

If B is given we can use Separation to define

$$R = \{(x,y) \in A \times B : P \wedge \forall z \in B : Q \supset y = z\}$$

where z is a variable not appearing in P and Q is the result of substituting z for all free occurences of y in P.

If A isn't finite and B isn't given then our current set of axioms don't suffice to prove the existence of R.

It's rarely necessary, but often convenient, to assume this is true, so we take it as an axiom, the Axiom of Replacement.

The natural numbers

We can *define* the natural numbers as lists all of whose elements are $\emptyset$.

Intuition: for each natural number there is precisely one such list whose length is that number.

So 0 is the empty list $()$.

x' is the list obtained by prepending $\emptyset$ to x.

$x + y$ is the concatenation of x and y.

$x \leq y$, $x \cdot y$, etc. can also be defined.

These definitions make our axioms from Elementary Arithmetic into theorems of Set Theory.

Elementary Arithmetic was semantically incomplete. What about Set Theory?

It must also be semantically incomplete!

Infinity

Elementary Arithmetic only had individual natural numbers, not sets of natural numbers. Does the Set Theory version have sets of natural numbers? Does it have the set of natural numbers?

All of our axioms so far–Elementary Sets, Extensionality, Separation, Power Sets, Union and Replacement–only allow us to construct finite sets from finite sets, so no!

We would like the set of natural numbers to exist, so we introduce an axiom, the Axiom of Infinity, which says that it does!