MAU11602

Lecture 22

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Gödel, again

Gödel proceeded a different direction from Tarski. We can use our encoding trick to replace the explicit self-reference in the sentence *No natural number encodes a proof of this statement* with an implicit self-reference.

In other words, there is a statement E asserting that for all n, n is not the encoding of a proof of E.

If E is a theorem then we can deduce theorems from it, one for each n, saying that n does not encode a proof of E.

Similarly, if $\neg E$ is a theorem then we can deduce the theorem that there is some n which encodes a proof of E.

If Peano arithmetic is syntactically complete then either E is a theorem or, if E is not a theorem, then $\neg E$ is a theorem.

In the first case E has a proof, encoded by some n, and there's a theorem saying that n encodes a proof and another theorem saying it doesn't, violating consistency.

In the second case there are theorems for each n saying n does not encode a proof, but also a theorem saying that some n does encode a proof.

ω consistency

The phenomenon which happens in the second case, where there's a property of natural numbers for which we can prove for each natural number that it holds, but also prove that there is a number for which it doesn't hold, doesn't violate syntactic completeness as defined earlier.

It's certainly weird though.

This is called ω *inconsistency*. A theory which is free from ω inconsistency is called ω *consistent*.

ω consistency is a stronger condition that ordinary consistency.

Gödel's argument shows that Peano arithmetic, no matter what axioms and rules of inference are chosen, must be ω inconsistent or incomplete.

This is known as Gödel's incompleteness theorem because people generally accept that Peano arithmetic, with the standard axioms and rules of inference, must be ω consistent, but this provable only in systems strictly stronger than Peano arithmetic, which is rather pointless.

Rosser's theorem

The theorem Gödel wanted to prove was that Peano arithmetic is inconsistent.

The theorem he probably wanted as a substitute was that Peano arithmetic is either inconsistent or incomplete.

The theorem he actually proved was that Peano arithmetic is either ω inconsistent or incomplete.

No one has succeeded in proving Peano arithmetic is inconsistent, but few people since Gödel have really tried.

A few years after Gödel's theorem, Rosser, mostly by accident, did manage to prove that arithmetic is either inconsistent or incomplete.

The key idea is to replace the statement E saying that for all n, n is not the encoding of a proof of E with a statement F saying that for each n if n encodes a proof of F then there is an $m < n$ which encodes a proof of $\neg F$.

Axioms and rules of inference

I've deliberately delayed introducing axioms or rules of inference for Peano arithmetic because the main theorems either don't mention them, like Tarski, or mostly don't care about the details of them, like Gödel and Rosser.

People do actually prove theorems in Peano arithmetic though, and for this you need axioms and rules of inference.

I will skip the purely logical ones and just give the ones for 0, S, $+$, $\cdot$, $<$, $>$, $\leq$, and $\geq$.

- $\forall x.\forall y.Sx = Sy \to x = y$
- $\forall x.\neg(Sx = 0)$
- $\forall x.0 + x = x$
- $\forall x.\forall y.S(x + y) = S(x) + y$
- $\forall x.0 \cdot x = 0$
- $\forall x.\forall y.Sx \cdot y = x \cdot y + y$

More axioms and rules of inference

- $\forall x. x \leq 0 \rightarrow x = 0$
- $\forall x. x = 0 \rightarrow x \leq 0$
- $\forall x. \forall y. x \leq y \vee y < x$
- $\forall x. \forall y. \neg(x \leq y \wedge y < x)$
- $\forall x. 0 \geq x \rightarrow 0 = x$
- $\forall x. 0 = x \rightarrow 0 \leq x$
- $\forall x. \forall y. x \geq y \vee y > x$
- $\forall x. \forall y. \neg(x \geq y \wedge y > x)$

There would be fewer of these if we used only one of the order relations, which we can do without loss of descriptive completeness.

In fact we sort of don't need any.

It's traditional to promote substitution from a derived rule of inference to a genuine rule of inference. If not you have to change all of the above axioms into rules of inference.

The other rule of inference we need to add is the principle of mathematical induction.

From $P[0/X]$ and $\forall X.(P \rightarrow P[SX/X])$ we can deduce $\forall X.P$.

Still more axioms and rules of inference

The rest depends on your version of first order logic. The usual choice is logic with existential presuppositions and without equality. In that case you need axioms about equality:

- $\forall x. x = x$
- $\forall x. \forall y. x = y \to y = x$
- $\forall x. \forall y. \forall z. x = y \wedge y = z \to x = z$

In a logic with equality but without existential presuppositions those are all theorems and so don't need to be added but you need axioms telling you that 0 exists and S, $+$, and $\cdot$ are total:

- $\exists w. w = 0$
- $\forall x. \exists w. w = Sx$
- $\forall x. \forall y. \exists w. w = x + y$
- $\forall x. \forall y. \exists w. w = x \cdot y$

Theorems

It is actually possible prove things in Peano arithmetic.

It's not pleasant to prove things in Peano arithmetic.

Commutativity of addition takes about 50-100 steps, depending on precisely which version of Peano arithmetic you use.

I'm not going to do this in lecture.

Associativity is somewhat easier. You can mostly just mimic the proof of associativity of `append`, being much more explicit about the steps.

The biggest problem in writing proofs is *not* using things you know to be true. For example, $\forall x . 0 + x = x$ and $\forall x . \forall y . S(x + y) = Sx + y$ are axioms but $\forall x . x + 0 = x$ and $\forall x . \forall y . S(x + y) = x + Sy$ are not. If you want to use them you have to prove them as theorems.

You might think "But I know they're also true!" but that's entirely irrelevant. We know that if Peano arithmetic is consistent there are true statements which are not theorems, so how do you know neither of these is one of them?

Automation

In Peano's time, and even in the time of Gödel, Tarski, and Rosser, you could only write proofs in Peano arithmetic by hand, and you could only check them by hand. Now you check proofs entirely automatically, *if you believe the proof checker is implemented correctly.*

This is not artificial intelligence. In fact the whole point is that no intelligence is required.

There are also *proof assistants*, programs which partially automate writing proofs. You generally have to give them lots of hints, but they do speed things up.

There are various projects to formalise parts of mathematics, i.e. to produce formal proofs of the main theorems.

For example, our earlier statement that there are infinitely many primes is actually a theorem.

Of course you need to trust the mechanical verification procedure *and* you need to check, or trust, that the formal statement of the theorem accurately reflects the familiar, informal, one.

Sets

Returning to a question from earlier, is Peano arithmetic descriptively complete, as a theory of the natural numbers?

This is a subjective question, so I can only tell you whether I think it's descriptively complete.

It's certainly more complete than it first looks. Even though it doesn't talk about sets and functions directly we can express properties of lots of sets and functions.

But not all sets and functions. Tarski's theorem gives us an example of an interesting set we can't describe in Peano arithmetic.

And if we want to talk about sets of sets of natural numbers, or functions from sets to sets, we're probably out of luck.

It would be convenient to have a theory of sets, not just of natural numbers but of sets in general.

It will still be incomplete. Mathematicians have mostly forgotten but Gödel's paper actually proved not only that Peano arithmetic is inconsistent or incomplete but also that the standard formulation of set theory is inconsistent or incomplete.