MAU11602

Lecture 16

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# Streams

Stacks are of finite size. Usually this is what we want, but sometimes you might want a (potentially) infinite one. We can do this!

For brevity I'll call a (potentially) infinite stack a stream.

Streams are best described in terms of `force` and `delay`.

A stream is a delayed expression which, when forced, returns either a unit, to signal the stream is empty, or an ordered pair, consisting of a head, an item, and a tail, another stream.

An empty stream is just a delayed unit.

Popping a stream is just forcing it.

Pushing an item onto a stream creates a delayed ordered pair with the item and the stream.

We can always make a stream from a stack by repeatedly pushing items onto an empty stream, but not every stream can be made into a stack.

Streams give us a way to represent infinite sequences when we will only need finitely many terms, but don't know in advance how many.

# Notions of completeness

There are several notions of completeness for formal systems, i.e. languages with a set of axioms and rules of inference:

- Semantic completeness: If a statement is true in all intended interpretations then it's a theorem.

This requires an interpretation or set of interpretations, of course.

- Syntactic completeness: Every statement or its negation is a theorem.

This requires negation, but doesn't otherwise require an interpretation. It does imply though that any two interpretations are essentially equivalent.

- Descriptive completeness: the language is capable of expressing everything we want to say about the subject it's meant to formalise.

This requires an interpretation and also an idea of the purpose of the system, but unlike the other two it doesn't depend on the choice of axioms and rules of inference.

It's possible to prove or disprove the semantic or syntactic completeness of a system, but not the descriptive completeness, so it's less studied, but without it the other two are kind of pointless.

# Examples

Classical zeroeth order logic is semantically complete but not syntactically complete, since the negation of a tautology needn't be a tautology.
Most people would regard it as descriptively complete, i.e. would say that it expresses everything it's meant to express.
That's mostly because the things it can't express are the subject of other theories of logic, e.g. first order logic, modal logic, temporal logic, etc.
Our language for arithmetic is neither semantically nor syntactically complete.
Without quantifiers it's certainly not descriptively complete, since we have no way to discuss divisibility. With quantifiers it probably is, depending on what you want to say.
First order logic won't be syntatically complete. We can't say whether it's semantically complete until we have rules of inference. For now our goal is to make it descriptively complete.

# Descriptive completeness of FOL

If $x = y$ and $y = z$ then $x = z$. This is the transitive property of equality. Is it part of logic?

If it's not then we will need to add it, or something equivalent, for every other theory, e.g. arithmetic, group theory, real analysis, etc.

If it is then we need an $=$ symbol in our language for logic or it won't be descriptively complete.

If $f(f(x, y), z) = f(x, f(y, z))$ for all $x$, $y$, and $z$ then $f(f(f(w, x), y), z) = f(w, f(x, f(y, z)))$ for all $w$, $x$, $y$, and $z$. In other words, if $f$ is an associative operation, i.e. can be applied to a list of three items either from left to right or right to left, then it can also be applied to a list of four items either from left to right or right to left.

Is this part of logic?

This one's more debatable. You could argue that this is algebra.

You can make it part of first or logic though, and if you do then you need function symbols for descriptive completeness.

# A language for FOL

I won't give a full formal grammar for first order logic but we'll include the following elements in the language.
- The logical constants $\top$ and $\bot$ and the Boolean operators $\wedge$, $\vee$, and $\rightarrow$
- The quantifiers $\forall$ and $\exists$
- The equality sign $=$
- Individual constants $a$, $b$, $c$, etc. and individual variables $w$, $x$, $y$, etc.
- Predicate variables $p$, $q$, $r$ etc.
- Functional variables $f$, $g$, $h$ etc.

We're only allowed to quantify over individual variables.
There are two types, boolean and the type of individuals in the domain (other).
Individual constants and variables have type other.
Two expressions of type other separated by a $=$ give a boolean expression.
We can also form expressions of type boolean and other by applying predicate or functional variables to arguments of type other, e.g. $p(a, x, y)$ or $f(a, b, x, y, z)$.

# Rules of inference for quantifiers

The rules of inference usually given for first order logic are the ones for zeroeth order logic plus the following introduction and elimination rules for $\forall$ and $\exists$.

$$\frac{\Gamma \vdash P[A/X]}{\Gamma \vdash \forall X.P} \qquad \frac{\Gamma,\, P[T/X] \vdash Q}{\Gamma,\, \forall X.P \vdash Q}$$

$$\frac{\Gamma \vdash P[T/X]}{\Gamma \vdash \exists X.P} \qquad \frac{\Gamma, P[A/X] \vdash Q}{\Gamma, \exists X.P \vdash Q}$$

Here $X$ represents some individual variable, $A$ represents some individual constant, $P$ represents a boolean expression, $T$ represents an other expression.
Bracket notation indicates capture avoiding substitution for free occurences of a variable, e.g. $P[T/X]$ is the result of substituting the expression $T$ for all free occurences of the variable $X$ in the expression $P$.
There are some restrictions on the application of these rules though. In the first and fourth rules the constant $A$ must not appear below the line, including in $\Gamma$. In the second and third rules no variables are allowed in the expression $T$.

# Rules of inference for quantifiers, continued

Compare the two possible rules of inference

$$\frac{\Gamma \vdash \forall X.P}{\Gamma \vdash P[T/X]} \qquad \frac{\Gamma\,,\ P[T/X] \vdash Q}{\Gamma\,,\ \forall X.P \vdash Q}$$

The first is a genuine elimination rule, we can use it to deduce a statement without a $\forall$ from one with a $\forall$. The second is what I called an elimination rule on the previous slide but it doesn't look like one. It is the usual way the rule is given though.

We can use the second to simulate the first though:

$$\frac{\dfrac{\dfrac{\Gamma\,,\ P[T/X] \vdash P[T/X]}{\Gamma\,,\ \forall X.P \vdash P[T/X]}}{\Gamma \vdash \forall X.P \to P[T/X]} \qquad \Gamma \vdash \forall X.P}{\Gamma \vdash P[T/X]}$$

So the first rule can be treated as a derived rule, like substitution. I'll use it, and its $\exists$ counterpart, as if they were ordinary rules.

# Rules of inference for equality

We also need rules for equality. We have the axiom $\Gamma \vdash T = T$ and the two rules

$$\frac{\Gamma, T_1 = T_2 \vdash P[T_1/X]}{\Gamma, T_1 = T_2 \vdash P[T_2/X]} \qquad \frac{\Gamma, T_1 = T_2 \vdash P[T_2/X]}{\Gamma, T_1 = T_2 \vdash P[T_1/X]}$$

These are enough to prove all the usual properties of equality, e.g. symmetry and transitivity.
Symmetry is mostly straighforward.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\vdash a = a}{a = b \vdash a = a}}{a = b \vdash b = a}}{\vdash a = b \to b = a}}{\vdash \forall y. a = y \to y = a}}{\vdash \forall x. \forall y. x = y \to y = x}$$

# Transitivity of equality

$$\dfrac{\dfrac{\dfrac{\dfrac{a = b \vdash a = b}{a = b, b = c \vdash a = b}}{a = b, b = c \vdash a = c}}{a = b \vdash b = c \to a = c}}{\vdash a = b \to b = c \to a = c}$$

Now substitute $a = b$, $b = c$, and $a = c$ for $p$, $q$, and $r$ in

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p \to q \to r \vdash p \to q \to r}{p \to q \to r\,,\, p \wedge q \vdash p \to q \to r} \qquad \dfrac{\dfrac{p \wedge q \vdash p \wedge q}{p \wedge q \vdash p}}{p \to q \to r\,,\, p \wedge q \vdash p}}{p \to q \to r\,,\, p \wedge q \vdash q \to r} \qquad \dfrac{\dfrac{p \wedge q \vdash p \wedge q}{p \wedge q \vdash q}}{p \to q \to r\,,\, p \wedge q \vdash q}}{p \to q \to r\,,\, p \wedge q \vdash r}}{p \to q \to r \vdash p \wedge q \to r}}{\vdash (p \to q \to r) \to (p \wedge q \to r)}$$

to get $\vdash (a = b \to b = c \to a = c) \to (a = b \wedge b = c \to a = c)$

# Transitivity of equality, continued

Splicing the two derivations on the previous slide into

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash (a = b \to b = c \to a = c) \qquad \vdash (a = b \to b = c \to a = c) \to (a = b \land b = c \to a = c)}{\vdash (a = b \land b = c \to a = c)}}{\vdash \forall z. a = b \land b = z \to a = z}}{\vdash \forall y. \forall z. a = y \land y = z \to a = z}}{\vdash \forall x. \forall y. \forall z. x = y \land y = z \to x = z}$$

gives a derivation of $\forall x. \forall y. \forall z. x = y \land y = z \to x = z$, which is therefore a theorem.

# Associativity

Here's most of a proof of

$$(\forall x.\forall y.\forall z.f(f(x,y),z) = f(x,f(y,z)))$$
$$\rightarrow (\forall w.\forall x.\forall y.\forall z.f(f(f(w,x),y),z) = f(w,f(x,f(y,z))))$$

$$\dfrac{\dfrac{\dfrac{\forall x.\forall y.\forall z.f(f(x,y),z) = f(x,f(y,z)) \vdash \forall x.\forall y.\forall z.f(f(x,y),z) = f(x,f(y,z))}{\forall x.\forall y.\forall z.f(f(x,y),z) = f(x,f(y,z)) \vdash \forall y.\forall z.f(f(f(a,b),y),z) = f(f(a,b),f(y,z))}}{\forall x.\forall y.\forall z.f(f(x,y),z) = f(x,f(y,z)) \vdash \forall z.f(f(f(a,b),c),z) = f(f(a,b),f(c,z))}}{\forall x.\forall y.\forall z.f(f(x,y),z) = f(x,f(y,z)) \vdash f(f(f(a,b),c),d) = f(f(a,b),f(c,d))}$$

$$\dfrac{\dfrac{\dfrac{\forall x.\forall y.\forall z.f(f(x,y),z) = f(x,f(y,z)) \vdash \forall x.\forall y.\forall z.f(f(x,y),z) = f(x,f(y,z))}{\forall x.\forall y.\forall z.f(f(x,y),z) = f(x,f(y,z)) \vdash \forall y.\forall z.f(f(a,y),z) = f(a,f(y,z))}}{\forall x.\forall y.\forall z.f(f(x,y),z) = f(x,f(y,z)) \vdash \forall z.f(f(a,b),z) = f(a,f(b,z))}}{\forall x.\forall y.\forall z.f(f(x,y),z) = f(x,f(y,z)) \vdash f(f(a,b),f(c,d)) = f(a,f(b,f(c,d)))}$$

# Associativity, continued

To complete the proof on the previous slide we need to splice the diagrams for

$$f(f(f(a,b),c),d) = f(f(a,b),f(c,d))$$

and

$$f(f(a,b),f(c,d)) = f(a,f(b,f(c,d))$$

into the one for transitivity of equality to get

$$f(f(f(a,b),c),d) = f(a,f(b,f(c,d)),$$

replace $a$, $b$, $c$ and $d$ by $w$, $x$, $y$, and $z$, each quantified by a $\forall$, and then finally use the introduction rule for $\rightarrow$ to make the context

$$\forall x.\forall y.\forall z.f(f(x,y),z) = f(x,f(y,z))$$

into the left hand side of an implication statement.