MAU11602

Lecture 10

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The empty type

The degenerate case of an n-tuple is the unit type, with only one value, `()`.

It's surprisingly useful. We used it together with tagged unions for implementing option types and booleans.

It can also be used to delay evaluation of expressions. Use the expression to define a function with an argument of unit type. The expression will be evaluated when the function is applied, but not when it's defined.

The degenerate case of tagged unions is the empty type, with no values.

This is less practically useful than the unit type, but still worth having around.

Some languages, like C, confuse the unit type and empty type, but $1 \neq 0$.

Zeroeth order logic

Zeroeth order logic, a.k.a the propositional calculus, deals with implication, conjunction and disjunction of statements *without analysing the content of those statements*.

Once we add elements like quantifiers we are in the realm of first order logic, a.k.a. the predicate calculus.

There is also higher order logic, but if you have sets you can generally avoid higher order logic.

Notation varies, but the most common choices are probably $\wedge$ for and, $\vee$ for or, and $\rightarrow$ for implies, all written in infix, with $\wedge$ having the highest precedence and $\rightarrow$ having the lowest. $\rightarrow$ is right-associative. Parentheses are used to override these conventions.

Constants for true and false may or may not be needed. These are much less standardised. If they're needed I will use $\top$ and $\bot$, respectively.

Prefix is also used, but not commonly.

Variables will be denoted by lower case Latin letters, starting with p.

Interpretations

The classical interpretation of zeroeth order logic is that variables represent propositions which are true or false, $\wedge$, $\vee$, and $\to$ mean and, or, and implies, respectively.

The or is an inclusive or. The implication is material rather than causative, so no relation between the statements is required beyond the second one being true whenever the first one is.

Example: If 42 is negative then 17 is positive or 17 is odd.

At least one of the statements 17 is positive or 17 is odd is true, so the full statement is true.

It doesn't matter that both are true, or that neither is caused by 42 being negative, which it of course isn't.

Closely related to the classical interpretation is an interpretation in terms of variables of type `bool`, with $\wedge$ and $\vee$ representing `andalso` and `orelse`, and $\top$ and $\bot$ representing `true` and `false`.

We can synthesise a $\to$ with statements of the form `if` p `then` q `else true`.

More interpretations

For any two real numbers $u < v$ we could interpret $\bot$ as u, $\top$ as v, $p \wedge q$ as $\min\{p, q\}$, $p \vee q$ as $\max\{p, q\}$, and $p \to q$ as $\max\{u + v - p, q\}$.

This would be a strange thing to do, except that this is how electronic computing works, with u and v being voltage levels, the precise values of which depend on the technology used.

There's also a set theory interpretation. Variables correspond to sets. $p \wedge q$ corresponds to the union of p and q, $p \vee q$ corresponds to the intersection of p and q, and $p \to q$ corresponds to the set theoretic difference of p and q, i.e. the set of members of p which are not members of q.

In this interpretation $\top$ corresponds to the empty set $\varnothing$ and $\bot$ *does not correspond to anything and should not be used*.

This is not an exhaustive list of interpretations. We will see at least one more later.

Proofs

For now, concentrate on the classical interpretation.

Certain combinations of statements are true no matter what the contents of the individual statements, provided they are all true or false, e.g. $p \to q \to p \land q$.

Such statements are called *tautologies*.

Checking that a statement is a (classical) tautology can be done via truth tables, but this is painful if the number of variables is large.

We'd like to be able to prove statements are tautologies without checking all possible combinations. For this we can use some simple rules.

The advantages are only felt for large expressions, but the examples in lecture and assignments will be small.

Rules of inference

If P and Q are tautologies then $P \wedge Q$ should be one. Conversely, if $P \wedge Q$ is a tautology then P and Q should each be tautologies.

I've written P and Q rather than p and q because we want to be able to apply this to expressions, not just variables, although variables are a kind of expression, so these rules also apply to variables.

If P and $P \rightarrow Q$ are tautologies then so is Q.

In mathematics we often make a temporary hypothesis, reason as if it's true, reach a conclusion, and then summarise by saying the conclusion implies the hypothesis.

This gives a mechanism for proving tautologies of the form $P \rightarrow Q$.

We might make further hypotheses within such an argument so we need a notion of a proof context: the set of hypotheses active at a given time.

Tautologies will be valid within an empty context, i.e. subject to no hypotheses, but intermediate steps may have context and those statements are only valid provisionally, subject to the assumptions in the context.

More rules of inference

We haven't considered $\vee$ yet.

If P is a tautology then $P \vee Q$ should be a tautology, and similarly if Q is a tautology.

We *cannot* safely assume that if $P \vee Q$ is a tautology then either P or Q must be a tautology. For example, $(p \rightarrow q) \vee (q \rightarrow p)$ is a (classical) tautology, but neither $p \rightarrow q$ nor $q \rightarrow p$ is a tautology.

We can say, though, that if $P \vee Q$ is true and R is true in any context where P is true and also in any context where Q is true then R is true.

This works also for conditional truths, i.e. if there are some additional assumptions beyond P or Q in the premises then R is true under those same assumptions, but without the assumption P or Q.

The other rules given earlier also apply to conditional truths, not just tautologies.

A formalisable proof

We haven't fully formalised this system, but we can already give proofs of some kind.

We know $p \to q \to p \land q$ is a tautology. Let's prove it.

Assume p. We're allowed to do this, but now p belongs to the context and all statements are conditional on p until we discharge this hypothesis.

Now assume q. Again, we're allowed to do this, but now these hypotheses p and q both belong to the context.

Derive $p \land q$. We can do this in our context, since p and q belong to the context and we're allowed to join (conditionally) true statements with an $\land$ to get a new (conditionally) true statement.

Discharge the hypothesis q. Now in a context with only the assumption p we have $q \to p \land q$.

$q \to p \land q$ is a more complicated statement than the $p \land q$ we had earlier, but it's true in a wider context, i.e. under weaker hypotheses.

Discharge the hypothesis p. Now in an empty context we have $p \to q \to p \land q$.

Since the context is empty we conclude that $p \to q \to p \land q$ is a tautology.

Formal proofs

Hopefully the argument on the previous slide convinced you $p \to q \to p \wedge q$ is a tautology.

It also shows that the given rules of inference allow us to prove it, i.e. that it is a theorem. That is not necessarily the same thing as being true!

Maybe our rules of inference are unsound, and allow us to prove statements which are not tautologies?

Maybe are rules are incomplete, and there are other tautologies we can't prove with them?

To answer these questions we need to formalise our arguments.

A convenient notation for conditionally valid statements gathers the context to the left of a $\vdash$, separated by commas, and the statement depending on these hypotheses to the right of the $\vdash$.

Proofs are traditionally written linearly, with each statement derived from some previous statements, usually leaving it to the reader to identify which ones. A more tree-like structure would make clearer which previous statements are used.

A diagrammatic proof

Combining the ideas from the previous slide we're led to the following diagram:

$$\dfrac{\dfrac{\dfrac{p,\, q \vdash p \qquad p,\, q \vdash q}{p,\, q \vdash p \wedge q}}{p \vdash q \rightarrow p \wedge q}}{\vdash p \rightarrow q \rightarrow p \wedge q}$$

This can be read from top down.
If we assume p then p is true. If we further assume q then p is still true.
The same holds with the roles of p and q reversed.
In a context where p and q are true we then have $p \wedge q$.
So in a context with only p we have $q \rightarrow p \wedge q$.
And therefore, in a global context, $p \rightarrow q \rightarrow p \wedge q$.
This type of reasoning, with or without the diagram, is called *natural deduction*.

Rules for diagrams

Diagrams are constructed one horizontal line at a time, with rules for how the statements on the top, and their context, relate to the statement on the bottom, and its context.

The rule for introducing hypotheses just says we can treat $\Gamma, P \vdash P$ as an axiom, i.e. a statement which needs no further justification. Also, adding a new hypothesis doesn't invalidate anything that we had under the old hypotheses:

$$\frac{\Gamma \vdash Q}{\Gamma, P \vdash Q}$$

Here Γ is a placeholder for some context, which may or may not be empty.

We have introduction and elimination rules for $\wedge$:

$$\frac{\Gamma \vdash P \qquad \Gamma \vdash Q}{\Gamma \vdash P \wedge Q} \quad \frac{\Gamma \vdash P \wedge Q}{\Gamma \vdash P} \quad \frac{\Gamma \vdash P \wedge Q}{\Gamma \vdash Q}$$

More rules for diagrams

There are also introduction and elimination rules for $\to$:

$$\frac{\Gamma, P \vdash Q}{\Gamma \vdash P \to Q} \qquad \frac{\Gamma \vdash P \qquad \Gamma \vdash P \to Q}{\Gamma \vdash Q}$$

The first rule corresponds to discharging a hypothesis.
Unlike the rules for $\wedge$, this rule changes the context.
There are also introduction and elimination rules for $\vee$:

$$\frac{\Gamma \vdash P}{\Gamma \vdash P \vee Q} \qquad \frac{\Gamma \vdash Q}{\Gamma \vdash P \vee Q} \qquad \frac{\Gamma \vdash P \vee Q \qquad \Gamma, P \vdash R \qquad \Gamma, Q \vdash R}{\Gamma \vdash R}$$

Again, the last rule changes the context.