MAU11602 Assignment 7
Due 2026-03-26
Solutions

1. Consider the following two statements:

$$\forall A.\exists B.\forall x.\,((\exists C.x \in C \wedge C \in A) \rightarrow (x \in B))$$

$$\forall A.\exists B.\forall x.\,(((\exists C.x \in C \wedge C \in A) \rightarrow (x \in B)) \wedge ((x \in B) \rightarrow (\exists C.x \in C \wedge C \in A)))$$

   (a) These are two different ways to formalise one of the axioms of set theory I introduced informally in lecture. Which axiom?

   Solution:

   The Axiom of Union. The second version says there's a B such that $x \in B$ if and only if $x \in C$ for some $C \in A$, which was our informal definition of $\bigcap A$, so this just says that $\bigcap A$ exists for all A.

   (b) It's possible to develop set theory using either of these statements as an axiom because we can get either from the other, using our other axioms and first order logic. In fact in one direction we only need first order logic. Which statement follows from the other by first order logic?

   Solution:

   The first follows from the second. In general we can deduce

$$\forall A.\exists B.\forall x.p(A, B, x) \rightarrow q(A, B, x)$$

   from

$$\forall A.\exists B.\forall x.\,(p(A, B, x) \rightarrow q(A, B, x)) \wedge (q(A, B, x) \rightarrow p(A, B, x))$$

   for any predicates p and q.

   (c) For the reverse direction we need to appeal to another axiom of set theory. Which one do we need?

   Solution:

   The first statement tells us there's a B such that every member of a member of A is a member of B, but it doesn't guarantee that those are the only members of B. We can get rid of any others though by using the Axiom of Separation to find a subset consisting of only those x which satisfy the Boolean expression

$$\exists C.x \in C \wedge C \in A.$$