MAU11602 Assignment 5
Due 2026-03-12
Solutions

1. Stacks and the append function were defined by

```
datatype 'a stack = Empty | Push of 'a * 'a stack
fun append (Empty, t) = t
  | append (Push (i, s), t) = Push (i, append (s, t))
```

We expect `append` to be associative, i.e. that

```
append (s, append (t, u)) = append (append (s, t), u)
```

Prove this using structural induction.

Solution: To prove

```
append (s, append (t, u)) = append (append (s, t), u)
```

by structural induction we need to show that this equation holds when `s = Empty` and that if it holds for some `s` then it holds with `s` replaced by `Push (i, s)`. The first of these is true because

```
append (Empty, append (t, u))
        = append (t, u)
        = append (append (Empty, t), u)
```

Here we've used the fact that `append (Empty, t) = t`, which is the first part of the definition of `append`. This holds for all stacks `t`, so it also holds with `t` replaced by `append (t, u)`, which we need in the first equation.

We prove

```
append (Push (i, s), append (t, u))
        = append (append (Push (i, s), t), u)
```

as follows

```
append (Push (i, s), append (t, u))
        = Push (i, append (s, append (t, u)))
        = Push (i, append (append (s, t), u))
        = append (Push (i, (append (s, t)), u)
        = append (append (Push (i, s), t), u)
```

The second equation is our induction hypothesis, which we're allowed to assume. The fourth equation follows from

```
Push (i, append (s, t)) = append (Push (i, s), t)
```

which is the second part of the definition of `append`. This holds for all stacks `s` and `t`, not just the particular stacks appearing in our induction hypothesis so we can replace `s` and `t` in this equation by any expressions of type `stack`. In particular, we can replace `t` by `append (t, u)`, which gives us the first equation above, and we can replace `t` by `u` and replace `s` by `append (s, t)`, which gives us the third equation above.