

MAU11602 Assignment 4

Due 2026-02-26

Solutions

1. (a) Write an expression with two free variables,  $d$  and  $n$ , expressing the fact that  $n$  is divisible by  $d$ , i.e. one which evaluates to true if and only if  $n$  is divisible by  $d$ . The language elements you have available are integer constants, the arithmetic operators  $\cdot$ ,  $+$ , and  $-$ , the arithmetic relations  $<$ ,  $\leq$ ,  $=$ ,  $\geq$ , and  $>$ , the symbols  $\wedge$ ,  $\vee$ ,  $\rightarrow$ ,  $\top$  and  $\perp$  from zeroth order logic, and the quantifiers  $\forall$  and  $\exists$  from first order logic.

*Solution:* There are multiple possibilities, but the obvious one is

$$\exists m. m \cdot d = n.$$

- (b) Write an expression with a single free variable  $x$  expressing the fact that  $n$  is prime. The language elements you can use are the same as in the previous part.

*Solution:* Again there are multiple possibilities but we need to express the fact that if  $d$  is a divisor then  $d = 1$ ,  $d = n$ ,  $d = -1$  or  $d = -n$ . The negative divisors need to be considered because we are dealing with integers rather than rational numbers. A further complication is that 1 satisfies the conditions above but is not prime. Also negative numbers aren't normally considered prime. So one way to express all of these conditions is

$$n > 1 \wedge \forall d. ((\exists m. m \cdot d = n) \rightarrow (d = 1 \vee d = n \vee d + 1 = 0 \vee d + n = 0)).$$

There are a number of other options though.

2. The functions

```
fun f2 0 = 0
  | f2 1 = 1
  | f2 n = if n > 0
            then f2 (n - 2) + f2 (n - 1)
            else f2 (n + 2) - f2 (n + 1)
fun h (0, (x, y)) = x
  | h (n, (x, y)) = if n > 0
```

```

        then h (n - 1, (y, x + y))
        else h (n + 1, (y - x, x))
fun f4 n = h (n, (0, 1))

```

appeared in lecture as ways to compute the  $n$ 'th Fibonacci number, but I never actually proved that `f2` and `f4` compute the same value if given the same argument. Prove this. You will probably want to use induction, but in order to do this you need to find an appropriate statement to apply induction to.

*Solution:* It's simplest to handle the cases of positive and negative  $n$  separately. For  $n$  positive, or more generally for  $n$  non-negative, we can prove

$$h(k, (f_2(n - k), f_2(n - k + 1))) = f_2(n)$$

for  $k$  in the range  $0 \leq k \leq n$  by induction on  $k$ . The base case,  $k = 0$ , is easy because

$$h(0, (f_2(n - 0), f_2(n - 0 + 1))) = h(0, f_2(n), f_2(n + 1)) = f_2(n).$$

For the inductive case we assume

$$h(k, (f_2(n - k), f_2(n - k + 1))) = f_2(n)$$

for a particular  $k$  in the range  $0 \leq k < n$  and prove the same statement with  $k$  replaced by  $k + 1$ .

$$\begin{aligned}
 & h(k + 1, (f_2(n - k - 1), f_2(n - k))) \\
 &= h(k, (f_2(n - k), f_2(n - k - 1) + f_2(n - k))) \\
 &= h(k, (f_2(n - k), f_2(n - k + 1))) \\
 &= f_2(n).
 \end{aligned}$$

For the first equation we use the definition of  $h$ . For the second we use the definition of  $f_2$  and for the third we use the induction hypothesis. Since both  $h$  and  $f_2$  are defined by cases a little of work is needed to verify that we are using the correct expression for each.  $k + 1 > 0$  so  $h(k + 1, (x, y))$  and  $h(k, (y, x + y))$  are interchangeable. Since  $k < n$  we have  $n - k + 1 > 1$  and so  $f_2(n - k + 1)$  and  $f_2(n - k - 1) + f_2(n - k)$  are interchangeable. We've therefore proved

$$h(k + 1, (f_2(n - k - 1), f_2(n - k))) = f_2(n),$$

which is indeed

$$h(k, (f_2(n-k), f_2(n-k+1))) = f_2(n)$$

with  $k$  replaced by  $k+1$ , so now we have the equation above for all  $k$  between 0 and  $n$ . Applying it to  $k=n$  gives

$$f_2(n) = h(n, (f_2(0), f_2(1))) = h(n, (0, 1)) = f_4(n).$$

The proof for negative  $n$  is similar, except that it proves the relation  $h(-k, (f_2(n+k), f_2(n+k+1))) = f_2(n)$  for  $k$  in the range  $0 \leq k \leq -n$  by induction on  $k$ .