

MAU11602 Assignment 3

Due 2026-02-19

Solutions

1. Commutativity of  $\vee$  is expressed by the tautology  $p \vee q \rightarrow q \vee p$ , which was proved in lecture. Associativity of  $\vee$  is expressed by the two tautologies  $p \vee (q \vee r) \rightarrow (p \vee q) \vee r$  and  $(p \vee q) \vee r \rightarrow p \vee (q \vee r)$ . Prove the first of these using a diagram where each step is justified by the inference rules.

*Solution:*

Splitting the diagram into two parts to make it fit on the page,

$$\frac{\frac{\frac{q \vdash q}{q \vdash p \vee q}}{q \vdash (p \vee q) \vee r} \quad \frac{r \vdash r}{r \vdash (p \vee q) \vee r}}{q \vee r \vdash q \vee r} \quad \frac{q \vee r, q \vdash (p \vee q) \vee r}{q \vee r \vdash (p \vee q) \vee r} \quad \frac{q \vee r, r \vdash (p \vee q) \vee r}{q \vee r \vdash (p \vee q) \vee r}$$

$$\frac{\frac{p \vdash p}{p \vdash p \vee q}}{p \vdash (p \vee q) \vee r} \quad \frac{q \vee r \vdash (p \vee q) \vee r}{p \vee (q \vee r), q \vee r \vdash (p \vee q) \vee r}}{p \vee (q \vee r) \vdash p \vee (q \vee r)} \quad \frac{p \vee (q \vee r), p \vdash (p \vee q) \vee r}{p \vee (q \vee r) \vdash (p \vee q) \vee r} \quad \frac{p \vee (q \vee r), q \vee r \vdash (p \vee q) \vee r}{\vdash p \vee (q \vee r) \rightarrow (p \vee q) \vee r}$$

2. Similarly, associativity of  $\wedge$  is expressed by the two tautologies  $p \wedge (q \wedge r) \rightarrow (p \wedge q) \wedge r$  and  $(p \wedge q) \wedge r \rightarrow p \wedge (q \wedge r)$ . Prove the first of these using a diagram where each step is justified by the inference rules.

*Solution:*

$$\frac{\frac{p \wedge (q \wedge r) \vdash p \wedge (q \wedge r)}{p \wedge (q \wedge r) \vdash p} \quad \frac{\frac{p \wedge (q \wedge r) \vdash p \wedge (q \wedge r)}{p \wedge (q \wedge r) \vdash q \wedge r}}{p \wedge (q \wedge r) \vdash q} \quad \frac{p \wedge (q \wedge r) \vdash p \wedge (q \wedge r)}{p \wedge (q \wedge r) \vdash q \wedge r}}{p \wedge (q \wedge r) \vdash p \wedge q} \quad \frac{p \wedge (q \wedge r) \vdash p \wedge (q \wedge r)}{p \wedge (q \wedge r) \vdash (p \wedge q) \wedge r} \quad \frac{p \wedge (q \wedge r) \vdash (p \wedge q) \wedge r}{\vdash p \wedge (q \wedge r) \rightarrow (p \wedge q) \wedge r}$$