MAU11602 Assignment 2
Due 2026-02-12
Solutions

1. Find all subexpressions of the expression below which contain at least one occurence of the variable `x` and indicate which of those occurences, if any, are free.

   ```
   * x let val y = x in + x let val x = 5 in - x y end end
   ```

   You should do this for all subexpressions, not just direct subexpressions, in other words for all nodes in the parse tree, not just the children of the root, although you aren't required to draw the parse tree.

   Solution:

   The expression

   ```
   * x let val y = x in + x let val x = 5 in - x y end end
     -             -      -
   ```

   is trivially a subexpression of itself. The free occurences of `x` in this expression are underlined.

   This expression has three direct subexpressions `*`, which has no occurences of `x` and which we can therefore ignore, `x`, which has a single occurrence, which is free, and

   ```
   let val y = x in + x let val x = 5 in - x y end end
               -      -
   ```

   where the two underlined occurences are free.

   From now on I won't mention subexpressions without an `x`. The other direct subexpressions of the expression above are another `x` by itself and

   ```
   + x let val x = 5 in - x y end
     -
   ```

   The only free occurence is the one underlined.

   Next we have another `x` by itself, again free, and

   ```
   let val x = 5 in - x y end
   ```

   which has no free occurences. Its relevant direct subexpressions are yet another free `x` and

   ```
   - x y
     -
   ```

   where the occurence is free, as indicated by the underlining.

Finally we have still one more `x` by itself, again free.

2. Mathematics has many kinds of expressions which bind variables. One is $\lim_{x \to e_1} e_2$, where e_1 and e_2 are numerical expressions and x is a variable, which is bound in the whole expression.

   Correctly substitute $x + z$ for y in the expression

   $$\lim_{x \to y} (x^2 + xy + y^2).$$

   Solution:

   The variable x occurs free in $x + z$ and so we have to perform and α conversion first. It doesn't matter what we rename x to as long as it's not x, y, or z. I'll take w. After α conversion we then have

   $$\lim_{w \to y} (w^2 + wy + y^2).$$

   Now we can safely substitute $x + z$ for all free occurences of y, which is all three of them. In fact we have to substitute $(x + z)$ for the second and third occurences though, so as not to disturb the precedence, since multiplication and exponentiation have higher precedence than addition:

   $$\lim_{w \to x+z} (w^2 + w(x + z) + (x + z)^2).$$