# MA 216
# Assignment 4
# Due 5 December 2007

1. Prove, in the $2 \times 2$ case, the assertion made in lecture that if $W$ is an invertible differentiable matrix valued function then

$$\frac{d}{dt}\det(W(t)) = \det(W(t))\,\mathrm{tr}(W'(t)W(t)^{-1})$$

*Solution:* If

$$W(t) = \begin{pmatrix} w_{11}(t) & w_{12}(t) \\ w_{21}(t) & w_{22}(t) \end{pmatrix}$$

then

$$\det(W(t)) = w_{11}(t)w_{22}(t) - w_{12}(t)w_{21}(t)$$

and

$$\frac{d}{dt}\det(W(t)) = w'_{11}(t)w_{22}(t) - w'_{12}(t)w_{21}(t) - w_{12}(t)w'_{21}(t) + w_{11}(t)w'_{22}(t).$$

Also,

$$W'(t) = \begin{pmatrix} w'_{11}(t) & w'_{12}(t) \\ w'_{21}(t) & w'_{22}(t) \end{pmatrix}$$

and

$$W(t)^{-1} = \frac{1}{\det W(t)} \begin{pmatrix} w_{22}(t) & -w_{12}(t) \\ -w_{21}(t) & w_{11}(t) \end{pmatrix}$$

so

$$W'(t)W(t)^{-1} = \frac{1}{\det W(t)} \begin{pmatrix} w'_{11}(t)w_{22}(t) - w'_{12}(t)w_{21}(t) & w_{11}(t)w'_{12}(t) - w'_{11}(t)w_{12}(t) \\ w'_{21}(t)w_{22}(t) - w_{21}(t)w'_{22}(t) & w_{11}(t)w'_{22}(t) - w_{12}(t)w'_{21}(t) \end{pmatrix}.$$

2. As shown in lecture, if we have a single nontrivial solution $x_1$ to the homogeneous equation

$$p(t)x''(t) + q(t)x'(t) + r(t)x(t) = 0$$

then we can, by the method of Wronski, find a second solution $x_2$, such that $x_1$ and $x_2$ form a basis for the set of solutions. Show that if $x_1$ and $x_2$ are such a basis then

$$W(t) = \begin{pmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{pmatrix} \begin{pmatrix} x_1(0) & x_2(0) \\ x_1'(0) & x_2'(0) \end{pmatrix}^{-1}$$

is the unique solution to the matrix initial value problem

$$W'(t) = A(t)W(t) \qquad W(0) = I$$

where

$$A(t) = \begin{pmatrix} 0 & 1 \\ -r(t)/p(t) & -q(t)/p(t) \end{pmatrix}.$$

*Solution:* Differentiate. Substitute.

3. Assuming the result of the previous exercise, find a solution to the *inhomogeneous* equation

$$p(t)x''(t) + q(t)x'(t) + r(t)x(t) = \varphi(t)$$

in terms of a pair $x_1, x_2$ of solutions to the homogeneous equation

$$p(t)x''(t) + q(t)x'(t) + r(t)x(t) = 0.$$

*Hint:* The equivalent first order system is

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = A(t) \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} 0 \\ \varphi(t)/p(t) \end{pmatrix}.$$

Use the general formula

$$\vec{x}(t) = W(t)\vec{x}(0) + \int_0^t W(t)W(s)^{-1}\vec{f}(s)\,ds,$$

for the solution to a first order inhomogeneous system

$$\vec{x}'(t) = A(t)\vec{x}(t) + \vec{f}(t).$$

*Solution:* Substituting,

$$\begin{aligned} W(t)\vec{x}(0) &= \begin{pmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{pmatrix} \begin{pmatrix} x_1(0) & x_2(0) \\ x_1'(0) & x_2'(0) \end{pmatrix}^{-1} \begin{pmatrix} x(0) \\ x'(0) \end{pmatrix} \\ &= \begin{pmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{pmatrix} \begin{pmatrix} \frac{x(0)x_2'(0)-x'(0)x_2(0)}{x_1(0)x_2'(0)-x_1'(0)x_2(0)} \\ \frac{x_1(0)x'(0)-x_1'(0)x(0)}{x_1(0)x_2'(0)-x_1'(0)x_2(0)} \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} W(t)W(s)^{-1} &= \begin{pmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{pmatrix} \begin{pmatrix} x_1(s) & x_2(s) \\ x_1'(s) & x_2'(s) \end{pmatrix}^{-1} \\ &= \frac{1}{x_1(s)x_2'(s)-x_1'(s)x_2(s)} \begin{pmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{pmatrix} \begin{pmatrix} x_2'(s) & -x_2(s) \\ -x_1'(s) & x_1(s) \end{pmatrix} \\ &= \begin{pmatrix} \frac{x_1(t)x_2'(s)-x_1'(s)x_2(t)}{x_1(s)x_2'(s)-x_1'(s)x_2(s)} & \frac{x_1(s)x_2(t)-x_1(t)x_2(s)}{x_1(s)x_2'(s)-x_1'(s)x_2(s)} \\ \frac{x_1'(t)x_2'(s)-x_1'(s)x_2'(t)}{x_1(s)x_2'(s)-x_1'(s)x_2(s)} & \frac{x_1(s)x_2'(t)-x_1'(t)x_2(s)}{x_1(s)x_2'(s)-x_1'(s)x_2(s)} \end{pmatrix}. \end{aligned}$$

Also,

$$\begin{aligned} x(t) = (\,1 \quad 0\,)\,\vec{x}(t) &= (\,1 \quad 0\,)\,W(t)\vec{x}(0) + \int_0^t (\,1 \quad 0\,)\,W(t)W(s)^{-1}\vec{f}(s)\,ds \\ &= \frac{x(0)x_2'(0)-x'(0)x_2(0)}{x_1(0)x_2'(0)-x_1'(0)x_2(0)}x_1(t) + \frac{x_1(0)x'(0)-x_1'(0)x(0)}{x_1(0)x_2'(0)-x_1'(0)x_2(0)}x_2(t) \\ &\quad + \int_0^t \frac{x_1(s)x_2(t)-x_1(t)x_2(s)}{x_1(s)x_2'(s)-x_1'(s)x_2(s)}\,\frac{\varphi(s)}{p(s)}\,ds \end{aligned}$$

4. The Chebyshev equation of order $n$ is

$$(1-t^2)x''(t) - tx'(t) + n^2x(t) = 0.$$

Special solutions are

$$x_1(t) = 1$$

when $n = 0$,

$$x_1(t) = t$$

when $n = 1$ and

$$x_1(t) = 2t^2 - 1$$

when $n = 2$. In each of these three cases, find the general solution. *Hint:* You will need to evaluate integrals of the form $\int R(t, \sqrt{1-t^2})\,dt$,

where $R$ is a rational function of two variables. Such integrals can always be evaluated in terms of elementary functions by means of the rationalising substution $t = 2u/(1+u^2)$ and partial fractions.
*Solution:* In each case

$$w(t) = \exp\left(\int_0^t \frac{s}{1-s^2}\,ds\right) = \frac{1}{\sqrt{1-t^2}}$$

and hence

$$\frac{x_2(t)}{x_1(t)} = \int_0^t \frac{ds}{x_1(s)^2\sqrt{1-s^2}} = \int_0^{t/(1+\sqrt{1-t^2})} \frac{2\,du}{x_1(2u/(1+u^2))^2(1+u^2)}.$$

If $n = 0$ then $x_1(t) = 1$ and

$$\frac{x_2(t)}{x_1(t)} = \int_0^{t/(1+\sqrt{1-t^2})} \frac{2\,du}{1+u^2} = 2\arctan(t/(1+\sqrt{1-t^2}) = \arcsin(t)$$

and

$$x_2(t) = 2\arctan(t/(1+\sqrt{1-t^2}).$$

If $n = 2$ then $x_1(t) = 2t^2 - 1$ and

$$\begin{aligned}
\frac{x_2(t)}{x_1(t)} &= 2\int_0^{t/(1+\sqrt{1-t^2})} \frac{(1+u^2)^3}{(u^4-6u^2+1)^2}\,du \\
&= \frac{1}{4}\int_0^{t/(1+\sqrt{1-t^2})} \left(\frac{2+\sqrt{2}}{(u-1-\sqrt{2})^2} + \frac{2-\sqrt{2}}{(u-1+\sqrt{2})^2}\right. \\
&\qquad \left. + \frac{2-\sqrt{2}}{(u+1-\sqrt{2})^2} + \frac{2+\sqrt{2}}{(u+1+\sqrt{2})^2}\right)\,du \\
&= -\frac{1}{4}\left[\frac{2+\sqrt{2}}{u-1-\sqrt{2}} + \frac{2-\sqrt{2}}{u-1+\sqrt{2}}\right. \\
&\qquad \left. + \frac{2-\sqrt{2}}{u+1-\sqrt{2}} + \frac{2+\sqrt{2}}{u+1+\sqrt{2}}\right]_{u=0}^{u=t/(1+\sqrt{1-t^2})} \\
&= \left[\frac{2u(1-u)^2}{u^4-6u^2+1}\right]_{u=0}^{u=t/(1+\sqrt{1-t^2})} \\
&= \frac{t\sqrt{1-t^2}}{2t^2-1}
\end{aligned}$$

and hence

$$x_2(t) = t\sqrt{1-t^2}.$$

If $n = 1$ then $x_1(t) = t$ and we have to be a bit more careful, since $x_1(0) = 0$. Nevertheless, we still have

$$\begin{aligned}\frac{x_2(t)}{x_1(t)} &= \frac{x_2(a)}{x_1(a)} + \int_a^{t/(1+\sqrt{1-t^2})} \frac{1+u^2}{2u^2}\, du \\ &= \frac{x_2(a)}{x_1(a)} + \left[\frac{u^2-1}{2u}\right]_{u=a}^{u=t/(1+\sqrt{1-t^2})}.\end{aligned}$$

By choosing the initial conditions for $x_2$ correctly we can arrange that the value at the lower limit cancels the $x_2(a)/x_1(a)$, leaving us with

$$\frac{x_2(t)}{x_1(t)} = -\frac{\sqrt{1-t^2}}{t}$$

and hence

$$x_2(t) = -\sqrt{1-t^2}.$$