MA 216
Assignment 3
Due 21 November 2007

1. (a) Given the Jordan decomposition

$$\begin{pmatrix} 17 & -9 \\ 25 & -13 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix},$$

solve the initial value problem

$$x'(t) = 17x(t) - 9y(t),$$
$$y'(t) = 25x(t) - 13y(t),$$
$$x(0) = \xi \qquad y(0) = \eta.$$

Solution:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \exp\left(t \begin{pmatrix} 17 & -9 \\ 25 & -13 \end{pmatrix}\right) \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

and

$$\begin{aligned} \exp\left(t \begin{pmatrix} 17 & -9 \\ 25 & -13 \end{pmatrix}\right) &= \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \exp\left(t \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}\right) \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} \exp(2t) & 0 \\ t\exp(2t) & \exp(2t) \end{pmatrix} \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix} \\ &= \begin{pmatrix} (1+15t)\exp(2t) & -9t\exp(2t) \\ 25t\exp(2t) & (1-15t)\exp(2t) \end{pmatrix} \end{aligned}$$

so

$$x(t) = (1+15t)\exp(2t)\xi - 9t\exp(2t)\eta$$

and

$$y(t) = 25t\exp(2t)\xi + (1-15t)\exp(2t)\eta.$$

(b) Given the Jordan decomposition

$$\begin{pmatrix} -22 & 15 \\ -39 & 26 \end{pmatrix} = \begin{pmatrix} 2+i & 2-i \\ 3+2i & 3-2i \end{pmatrix} \begin{pmatrix} 2+3i & 0 \\ 0 & 2-3i \end{pmatrix} \begin{pmatrix} (2+3i)/2 & (-1-2i)/2 \\ (2-3i)/2 & (-1+2i)/2 \end{pmatrix},$$

solve the initial value problem

$$x'(t) = -22x(t) + 15y(t),$$

$$y'(t) = -39x(t) + 26y(t),$$

$$x(0) = \xi \qquad y(0) = \eta.$$

Solution:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \exp(tA) \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

where

$$A = \begin{pmatrix} -22 & 15 \\ -39 & 26 \end{pmatrix} = PJP^{-1},$$

$$P = \begin{pmatrix} 2+i & 2-i \\ 3+2i & 3-2i \end{pmatrix},$$

$$J = \begin{pmatrix} 2+3i & 0 \\ 0 & 2-3i \end{pmatrix}$$

and

$$P^{-1} = \begin{pmatrix} (2+3i)/2 & (-1-2i)/2 \\ (2-3i)/2 & (-1+2i)/2 \end{pmatrix}.$$

Then

$$\begin{aligned} \exp(tA) &= P\exp(tJ)P^{-1} \\ &= P \begin{pmatrix} \exp(2t)(\cos(3t) + i\sin(3t)) & 0 \\ 0 & \exp(2t)(\cos(3t) - i\sin(3t) \end{pmatrix} P^{-1} \\ &= \begin{pmatrix} \exp(2t)(3\cos(3t) - 24\sin(3t)) & 15\exp(2t)\sin(3t) \\ -39\exp(2t)\sin(3t) & \exp(2t)(3\cos(3t) + 24\sin(3t)) \end{pmatrix} \end{aligned}$$

so

$$x(t) = \exp(2t)(3\cos(3t) - 24\sin(3t)\xi + 15\exp(2t)\sin(3t)\eta$$

and

$$y(t) = -39\exp(2t)\sin(3t)\xi + \exp(2t)(3\cos(3t) + 24\sin(3t))\eta.$$

2. (a) Find a basis for the space of solutions to the linear homogeneous differential equation

$$x''''(t) - x(t) = 0.$$

Solution: The characteristic polynomial of the coefficient matrix of the corresponding first order system can be read off from the equation:

$$p_A(\lambda) = \lambda^4 - 1 = (\lambda - 1)(\lambda + 1)(\lambda - i)(\lambda + i).$$

From this factorisation we obtain the complex basis

$$\{\exp(t), \exp(-t), \exp(it), \exp(-it)\}.$$

The real basis

$$\{\exp(t), \exp(-t), \cos(t), \sin(t)\}$$

may be more convenient however.

(b) Find all solutions of the linear inhomogenous differential equation

$$x''''(t) - x(t) = t.$$

Solution: In the notation of lecture, $f(t) = t$ and the polynomials p and q are

$$p(s) = s^4 - 1 = (s - 1)(s + 1)(s - i)(s + i)$$

and

$$q(s) = s^2$$

since the equation is of the form

$$p\left(\frac{d}{dt}\right) x = f$$

and

$$q\left(\frac{d}{dt}\right) f = 0.$$

The x we are looking for therefore satisfies the *sixth order* homogeneous equation

$$p\left(\frac{d}{dt}\right)q\left(\frac{d}{dt}\right) = 0$$

for which a basis for the solution space is

$$\{\exp(t), \exp(-t), \cos(t), \sin(t), t, 1\}.$$

The general solution to the sixth order homogeneous equation is therefore

$$x(t) = \alpha\exp(t) + \beta\exp(-t) + \gamma\cos(t) + \delta\sin(t) + \epsilon t + \zeta.$$

Substituting in the original fourth order inhomogeneous equation

$$x''''(t) - x(t) = t$$

we see that $\epsilon = -1$ and $\zeta = 0$. The other coefficients, α, β, γ and δ, could take any values, so

$$x(t) = \alpha\exp(t) + \beta\exp(-t) + \gamma\cos(t) + \delta\sin(t) - t.$$

It is possible to express this solution in terms of $x(0)$, $x'(0)$, $x''(0)$ and $x'''(0)$ rather than α, β, γ and δ, but this is not required.

3. Solve the initial value problem for the forced harmonic oscillator

$$mx''(t) + 2\gamma x'(t) + kx(t) = C\cos(\omega t - \varphi)$$

in the exceptional case

$$\gamma = 0 \qquad \omega^2 = k/m.$$

Solution: It is convenient to rewrite the equation in the form

$$x''(t) + \omega^2 x(t) = \frac{C}{m}\cos(\varphi)\cos(\omega t) + \frac{C}{m}\sin(\varphi)\sin(\omega t).$$

In this case

$$p(s) = q(s) = s^2 + \omega^2$$

and hence

$$p(s)q(s) = (s^2 + \omega^2)^2 = (s - i\omega)^2(s + i\omega)^2$$

so we have two roots of multiplicity two and the solution is a linear combination of

$$\{\exp(i\omega t), \exp(-i\omega t), t\exp(i\omega t), t\exp(-i\omega t)\}$$

or

$$\{\cos(\omega t), \sin(\omega t), t\cos(\omega t), t\sin(\omega t)\}.$$

Differating

$$x(t) = a\cos(\omega t) + b\sin(\omega t) + At\cos(\omega t) + Bt\sin(\omega t)$$

twice, we see that

$$x'(t) = (A + b\omega)\cos(\omega t) + (B - a\omega)\sin(\omega t) + B\omega t\cos(\omega t) - A\omega t\sin(\omega t)$$

and

$$x''(t) = (2B\omega - a\omega^2)\cos(\omega t) + (-2A\omega + b\omega^2)\sin(\omega t) - A\omega^2 t\cos(\omega t) - B\omega^2 t\sin(\omega t).$$

Substituting into the original equation

$$x''(t) + \omega^2 x(t) = \frac{C}{m}\cos(\varphi)\cos(\omega t) + \frac{C}{m}\sin(\varphi)\sin(\omega t),$$

we see that

$$2B\omega\cos(\omega t) - 2A\omega\sin(\omega t) = \frac{C}{m}\cos(\varphi)\cos(\omega t) + \frac{C}{m}\sin(\varphi)\sin(\omega t)$$

and hence that

$$A = -\frac{C}{2m\omega}\sin(\varphi) \qquad B = \frac{C}{2m\omega}\cos(\varphi)$$

while a and b could take any values. Then

$$x(t) = a\cos(\omega t) + b\sin(\omega t) + \frac{Ct}{2m\omega}\left(-\sin(\varphi)\cos(\omega t) + \cos(\varphi)\sin(\omega t)\right)$$

or

$$x(t) = a\cos(\omega t) + b\sin(\omega t) + \frac{C}{2m\omega}t\sin(\omega t - \varphi).$$

Differentiating and substituting $t = 0$,

$$x(0) = a \quad x'(0) = b\omega - \frac{C}{2m\omega}\sin(\varphi)$$

so

$$a = x(0) \quad b = \frac{1}{\omega}\left(x'(0) + \frac{C}{2m\omega}\sin(\varphi)\right).$$

The solution to the initial value problem is therefore

$$x(t) = x(0)\cos(\omega t) + \left(\frac{x'(0)}{\omega} + \frac{C}{2k}\sin(\varphi)\right)\sin(\omega t) + \frac{C}{2m\omega}t\sin(\omega t - \varphi)$$