MA 216
Assignment 2
Due 7 November 2007

1. Rewrite each of the following higher order scalar equations as a first order system.

   (a) Hook's law

   $$mx''(t) + kx(t) = 0$$

   Solution:

   $$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -k/m & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

   (b) Bessel's equation

   $$t^2 x''(t) + tx'(t) + (t^2 - \nu^2)x(t) = 0,$$

   Solution:

   $$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \frac{\nu^2}{t^2} - 1 & -\frac{1}{t} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

   (c) The Laguerre Differential Equation

   $$tx''(t) + (1-t)x'(t) + \nu x(t) = 0.$$

   Solution:

   $$\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{\nu}{t} & 1 - \frac{1}{t} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

2. It was shown in lecture that if $AB = BA$ then each of the following equations is satisfied for all real t:

$$\exp(tA + tB) = \exp(tA)\exp(tB),$$

$$\exp(tA + tB) = \exp(tB)\exp(tA)$$

and

$$\exp(tA)\exp(tB) = \exp(tB)\exp(tA).$$

Prove that, conversely, if any of these three equations is satisfied for all t then $AB = BA$.
Hint: Differentiate twice.
Solution: Set

$$f(t) = \exp(tA + tB)$$
$$g(t) = \exp(tA)\exp(tB),$$
$$h(t) = \exp(tB)\exp(tA).$$

Then, using the differential equation and the product rule,

$$f'(t) = (A + B)f(t),$$

$$g'(t) = Ag(t) + g(t)B$$

and

$$h'(t) = Bh(t) + h(t)A.$$

Differentiating again,

$$f''(t) = (A + B)^2 f(t),$$

$$g''(t) = A^2 g(t) + 2Ag(t)B + g(t)B^2$$

and

$$h''(t) = B^2 h(t) + 2Bh(t)A + h(t)A^2.$$

Evaluating at 0 using the fact that $f(0) = g(0) = h(0)$,

$$f''(0) = A^2 + AB + BA + B^2,$$

$$g''(0) = A^2 + 2AB + B^2$$

and

$$h''(0) = A^2 + 2BA + B^2.$$

Thus

$$AB - BA = f''(0) - h''(0) = g''(0) - f''(0) = \frac{1}{2}(g''(0) - h''(0))$$

is zero if $f = g$ or $g = h$ or $f = h$.

3. Compute the following matrix exponentials:

   (a)

$$\exp\left(t\begin{pmatrix} 13 & -5 \\ 30 & -12 \end{pmatrix}\right)$$

Solution:

$$\mathrm{tr}(A) = 1 \qquad \det(A) = -6$$

$$\mu = \frac{1}{2}\,\mathrm{tr}(A) = \frac{1}{2} \qquad \Delta = \mathrm{tr}(A)^2 - 4\det(A) = 25 > 0$$

$$\exp(tA) = I\cosh(t\sqrt{\Delta}/2) + (A - \mu I)\frac{\sinh(t\sqrt{\Delta}/2)}{\sqrt{\Delta}/2}$$

$$\exp\left(t\begin{pmatrix} 13 & -5 \\ 30 & -12 \end{pmatrix}\right) =$$

$$\exp(t/2)\begin{pmatrix} \cosh(5t/2) + 5\sinh(5t/2) & -2\sinh(5t/2) \\ 12\sinh(5t/2) & \cosh(5t/2) - 5\sinh(5t/2) \end{pmatrix}$$

$$= \begin{pmatrix} 3\exp(3t) - 2\exp(-2t) & -\exp(3t) + \exp(-2t) \\ 6\exp(3t) - 6\exp(-2t) & -2\exp(3t) + 3\exp(-2t) \end{pmatrix}$$

   (b)

$$\exp\left(t\begin{pmatrix} 1 & 1 \\ -4 & 5 \end{pmatrix}\right)$$

Solution:

$$\mathrm{tr}(A) = 6 \qquad \det(A) = 9$$

$$\mu = \frac{1}{2}\,\mathrm{tr}(A) = 3 \qquad \Delta = \mathrm{tr}(A)^2 - 4\det(A) = 0$$

$$\exp(tA) = I + (A - \mu I)t$$

$$\exp\left(t\begin{pmatrix} 1 & 1 \\ -4 & 5 \end{pmatrix}\right) = \exp(3t)\begin{pmatrix} 1 - 2t & t \\ -4t & 1 + 2t \end{pmatrix}$$

(c)

$$\exp\left(t\begin{pmatrix}-2 & 2\\ -13 & 8\end{pmatrix}\right)$$

Solution:

$$\mathrm{tr}(A) = 6 \qquad \det(A) = 10$$

$$\mu = \frac{1}{2}\,\mathrm{tr}(A) = 3 \qquad \Delta = \mathrm{tr}(A)^2 - 4\det(A) = -4 < 0$$

$$\exp(tA) = I\cos(t\sqrt{-\Delta}/2) + (A - \mu I)\frac{\sin(t\sqrt{-\Delta}/2)}{\sqrt{-\Delta}/2}$$

$$\exp\left(t\begin{pmatrix}-2 & 2\\ -13 & 8\end{pmatrix}\right) = \exp(3t)\begin{pmatrix}\cos(t) - 5\sin(t) & 2\sin(t)\\ -13\sin(t) & \cos(t) - 5\sin(t)\end{pmatrix}$$

4. Find the solution to the initial value problem

$$x(0) = \xi_1 \qquad x'(0) = \xi_2$$

for the general linear constant coefficient second order scalar equation

$$c_2x''(t) + c_1x'(t) + c_0x(t) = 0$$

by reducing it to a first order system.
Solution: The corresponding first order system is

$$\begin{pmatrix}x_1'(t)\\ x_2'(t)\end{pmatrix} = \begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}$$

where

$$A = \begin{pmatrix}0 & 1\\ -c_0/c_2 & -c_1/c_2\end{pmatrix}$$

and, as usual, $x_1 = x$ and $x_2 = x'$. The solution is

$$\begin{pmatrix}x(t)\\ x'(t)\end{pmatrix} = \begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} = \exp(tA)\begin{pmatrix}x_1(0)\\ x_2(0)\end{pmatrix} = \exp(tA)\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix}.$$

The form of the exponential depends on

$$\Delta = \frac{c_1^2 - 4c_0c_2}{c_2^2}.$$

If $\Delta < 0$ then $\exp(tA)$ is

$$\exp\left(-\frac{c_1 t}{2c_2}\right)\begin{pmatrix} \cos(t\sqrt{-\Delta}/2) + \frac{c_1}{c_2\sqrt{-\Delta}}\sin(t\sqrt{-\Delta}/2) & \frac{2}{\sqrt{-\Delta}}\sin(t\sqrt{-\Delta}/2) \\ -\frac{2c_0}{c_2\sqrt{-\Delta}}\sin(t\sqrt{-\Delta}/2) & \cos(t\sqrt{-\Delta}/2) - \frac{c_1}{c_2\sqrt{-\Delta}}\sin(t\sqrt{-\Delta}/2) \end{pmatrix}$$

and

$$x(t) = \exp\left(-\frac{c_1 t}{2c_2}\right)\left[\left(\cos(t\sqrt{-\Delta}/2) + \frac{c_1}{c_2\sqrt{-\Delta}}\sin(t\sqrt{-\Delta}/2)\right)\xi_1 + \frac{2}{\sqrt{-\Delta}}\sin(t\sqrt{-\Delta}/2))\xi_2\right].$$

If $\Delta = 0$ then $\exp(tA)$ is

$$\exp\left(-\frac{c_1 t}{2c_2}\right)\begin{pmatrix} 1 + \frac{c_1 t}{2c_2} & t \\ -\frac{c_0 t}{c_2} & 1 - \frac{c_1 t}{2c_2} \end{pmatrix}$$

and

$$x(t) = \exp\left(-\frac{c_1 t}{2c_2}\right)\left[\left(1 + \frac{c_1 t}{2c_2}\right)\xi_1 + t\xi_2\right]$$

If $\Delta < 0$ then $\exp(tA)$ is

$$\exp\left(-\frac{c_1 t}{2c_2}\right)\begin{pmatrix} \cosh(t\sqrt{\Delta}/2) + \frac{c_1}{c_2\sqrt{\Delta}}\sinh(t\sqrt{\Delta}/2) & \frac{2}{\sqrt{\Delta}}\sinh(t\sqrt{\Delta}/2) \\ -\frac{2c_0}{c_2\sqrt{\Delta}}\sinh(t\sqrt{\Delta}/2) & \cosh(t\sqrt{\Delta}/2) - \frac{c_1}{c_2\sqrt{\Delta}}\sinh(t\sqrt{\Delta}/2) \end{pmatrix}.$$

and

$$x(t) = \exp\left(-\frac{c_1 t}{2c_2}\right)\left[\left(\cosh(t\sqrt{\Delta}/2) + \frac{c_1}{c_2\sqrt{\Delta}}\sinh(t\sqrt{\Delta}/2)\right)\xi_1 + \frac{2}{\sqrt{\Delta}}\sinh(t\sqrt{\Delta}/2))\xi_2\right].$$