MA 216
Assignment 1
Due 23 October 2007

1. For each of the following, say whether it is a scalar equation or a system, what its order is, whether it is linear or nonlinear, and, if linear, whether it is homogeneous or inhomogeneous.

   (a) Bessel's equation:

   $$t^2 x''(t) + t x'(t) + (t^2 - \nu^2) x(t) = 0,$$

   Solution: A second order linear homogeneous scalar equation.

   (b) Jacobi's modular equation:

   $$-3k''(t)^2 + 2k'(t)k'''(t) + \left(\frac{1 + k(t)^2}{k(t) - k(t)^3}\right)^2 k'(t)^4 - \left(\frac{1 + t^2}{t - t^3}\right)^2 k'(t)^2 = 0,$$

   Solution: A third order nonlinear scalar equation.

   (c) The equation of motion of a pendulum:

   $$l\theta''(t) + g \sin(\theta(t)) = 0$$

   Solution: A second order nonlinear scalar equation.

   (d) The Lorenz equations

   $$x'(t) = \sigma \cdot (y(t) - x(t))$$

$$y'(t) = x(t)(\rho - z(t)) - y(t)$$
$$z'(t) = x(t)y(t) - \beta z(t),$$

Solution: A first order nonlinear system.

(e) The mortgage repayment equation:

$$x'(t) = rx(t) - A,$$

Solution: A first order linear inhomogeneous scalar equation.

(f) The circular motion equations:

$$x'(t) = -y(t)$$
$$y'(t) = x(t).$$

Solution: A first order linear homogeneous system.

2. (a) Prove that

$$E(t) = x(t)^2 + y(t)^2$$

is an invariant of the system

$$x'(t) = -y(t)$$
$$y'(t) = x(t).$$

and use this fact to obtain bounds on $x(t)$ and $y(t)$ in terms of $x(0)$ and $y(0)$.
Solution: Differentiating,

$$E'(t) = 2x(t)x'(t) + 2y(t)y'(t).$$

Since x and y are assumed to satisfy the system,

$$E'(t) = 0$$

and E is therefore constant. From

$$x(t)^2 \leq x(t)^2 + y(t)^2 = x(0)^2 + y(0)^2$$

and

$$y(t)^2 \leq x(t)^2 + y(t)^2 = x(0)^2 + y(0)^2$$

it follows that

$$|x(t)| \leq \sqrt{x(0)^2 + y(0)^2}$$

and

$$|x(t)| \leq \sqrt{x(0)^2 + y(0)^2}.$$

(b) Prove that

$$E(t) = x(t)y(t)$$

is an invariant of the system

$$x'(t) = -x(t)$$

$$y'(t) = y(t).$$

This fact does not provide any useful bounds on $x(t)$ and $y(t)$. What makes this example different from the previous one?
Solution: In this case

$$E'(t) = x(t)y'(t) + x'(t)y(t) = 0$$

for solutions of the system. The problem in this case is that knowing E provides almost[1] no information on the possible values of $x(t)$ and $y(t)$.

3. (a) The Euler equations for a rigid body

$$I_1\Omega_1'(t) = (I_2 - I_3)\Omega_2(t)\Omega_3(t)$$

$$I_2\Omega_2'(t) = (I_3 - I_1)\Omega_3(t)\Omega_1(t)$$

$$I_3\Omega_3'(t) = (I_1 - I_2)\Omega_1(t)\Omega_2(t)$$

which were shown in lecture to possess an invariant

$$E(t) = \frac{1}{2}I_1\Omega_1(t)^2 + \frac{1}{2}I_2\Omega_2(t)^2 + \frac{1}{2}I_3\Omega_3(t)^2$$

[1] Knowing $E \neq 0$ does tell us that $x(t) \neq 0$ and $y(t) \neq 0$, but this does not give any bounds.

have another invariant, which is also quadratic in the variables Ω_1, Ω_2 and Ω_3. Find it.
Solution: The other invariant is

$$M(t) = \frac{1}{2} I_1^2 \Omega_1(t)^2 + \frac{1}{2} I_2^2 \Omega_2(t)^2 + \frac{1}{2} I_3^2 \Omega_3(t)^2.$$

Any multiple of this will work as well or, more generally, any combination of E and M.

(b) The equations are unchanged by cyclically permuting the indices 1, 2 and 3. The transformation

$$\begin{aligned}\tilde{\Omega}_1(t) &= \Omega_2(t)\\ \tilde{\Omega}_2(t) &= \Omega_3(t)\\ \tilde{\Omega}_3(t) &= \Omega_1(t)\end{aligned}$$

is not, however, a symmetry of the system. Why not?
Solution: The functions $\tilde{\Omega}_1$, $\tilde{\Omega}_2$ and $\tilde{\Omega}_3$ satisfy the system

$$\begin{aligned}I_2 \tilde{\Omega}_1'(t) &= (I_3 - I_1)\tilde{\Omega}_2(t)\tilde{\Omega}_3(t)\\ I_3 \tilde{\Omega}_2'(t) &= (I_1 - I_2)\tilde{\Omega}_3(t)\tilde{\Omega}_1(t)\\ I_1 \tilde{\Omega}_3'(t) &= (I_2 - I_3)\tilde{\Omega}_1(t)\tilde{\Omega}_2(t)\end{aligned}$$

which is not, except in the special case $I_1 = I_2 = I_3$, the same as the original system. It is a system of the same type, but not the same system.

4. (a) Prove that any twice differentiable solution of the equation

$$\frac{1}{2} m x'(t)^2 + \frac{1}{2} k x(t)^2 = E$$

is either constant $\pm\sqrt{E/2k}$ or is a solution of the differential equation

$$m x''(t) + k x(t) = 0.$$

Solution: Let $A = \sqrt{2E/k}$. Since x is twice differentiable, we may differentiate the differential equation once to obtain

$$x'(t)(m x''(t) + k x(t)) = 0.$$

so the differential equation $mx''(t) + kx(t) = 0$ is satisfied, except possibly at points where $x'(t) = 0$, where either $x = A$ or $x = -A$. In other words, each $t \in \mathbf{R}$ belongs to exactly one of the following sets

$$S_1 = \{t \in \mathbf{R}\colon mx''(t) + kx(t) = 0\},$$
$$S_2 = \{t \in \mathbf{R}\colon mx''(t) + kx(t) \neq 0, x(t) = A\},$$
$$S_3 = \{t \in \mathbf{R}\colon mx''(t) + kx(t) \neq 0, x(t) = -A\}.$$

What we need to show is that all t belong to *the same* set S_j. This is, I have to admit, much harder to do than I thought when I assigned the problem. Life would be much simpler if I had written "twice continuously differentiable" in place of "twice differentiable," but it's too late to do anything about that now. Even though we don't know that x'' is continuous, it does at least have the intermediate value property, as a consequence of the following lemma, applied to $f = x'$.

Lemma 1 *If f is differentiable in the interval $[a, b]$ and either $f'(a) < y < f'(b)$ or $f'(b) < y < f'(a)$ then there is a $t \in (a, b)$ such that $f'(t) = y$.*

The proof of the lemma will be given later.

The main idea is to use the following lemma, which essentially says that the real line is connected.

Lemma 2 *Suppose S_1, S_2, ..., S_n are subsets of $\mathbf{R}$ such that*

- $S_1 \cup S_2 \cup \cdots \cup S_n = \mathbf{R}$,
- *if $t \in S_j$ then there is a $\delta > 0$ such that $(t - \delta, t + \delta) \subset S_j$, and*
- *if $j \neq k$ then $S_j \cap S_k = \emptyset$.*

Then there is a $j \in \{1, 2, \ldots, n\}$ such that $S_j = \mathbf{R}$.

In fact the last of the three conditions is not needed, but it is satisfied in our case and it makes the proof of the lemma slightly easier. The proof will be given later.

Assuming Lemma 2, what we need to show is that the three sets above have the property that if $t \in S_j$ then there is a $\delta > 0$ such that $(t - \delta, t + \delta) \subset S_j$. This will be proved separately for each j.

Suppose now that $t \in S_1$. From the differential equation

$$\frac{1}{2}mx'(t)^2 + \frac{1}{2}kx(t)^2 = E$$

it follows that $x(t) \in [-A, A]$. Note that x is twice differentiable and therefore continuous, a fact which we will need several times. If $-A < x(t) < A$ then, by continuity, there is a $\delta > 0$ such that $-A < x(s) < A$ for all $s \in (t-\delta, t-\delta)$. No such s can be in either S_2 or S_3, so $(t-\delta, t-\delta) \subset S_1$. If $x(t) = A$ then $x'(t) = 0$ and $x''(t) = -kA/m < 0$, so x has a strict local maximum at t. In other words, there is a $\delta > 0$ such that $x(s) < x(t) = A$ for $t - \delta_1 < s < t$ or $t < s < t + \delta_1$. By continuity there is a δ_2 such that $x(s) > -A$ for $t - \delta_2 < s < t + \delta_2$. Setting $\delta = \min\{\delta_1, \delta_2\}$ we have $-A < x(s) < A$ for $t - \delta < s < t$ or $t < s < t + \delta$. As we already saw, $-A < x(s) < A$ implies $s \in S_1$, so $(t-\delta, t-\delta) \subset S_1$. If $x(t) = -A$ we can apply the same argument with some of the signs reversed, and we again find that $(t-\delta, t-\delta) \subset S_1$. Since these three cases exhaust all the possibilities for $x(t)$ we see that if $x(t) \in S_1$ then $(t-\delta, t-\delta) \subset S_1$.

I claim that if $t \in S_2$ then $x''(0) = 0$. From the definition of S_2,

$$z = mx''(t) + kx(t) \neq 0.$$

It is clear that $x''(t) \leq 0$, for otherwise x would have a strict local minimum at t and would therefore take values greater than A somewhere. Assume that $x''(t) < 0$. Then, as in the previous paragraph, there is a $\delta_1 > 0$ such that $(t-\delta_1, t) \cup (t, t+\delta_1) \subset S_1$. Set $\epsilon = \frac{|z|}{2k}$. By the continuity of x there is a $\delta_2 > 0$ such that $s \in (t-\delta_1, t+\delta_1)$ implies $|x(s) - A| \leq \epsilon$. Set $\delta = \min(\delta_1, \delta_2)$ and choose an $s \neq t$ from $(t-\delta, t-\delta)$. Then $s \in S_1$, so $mx''(s) + kx(s) = 0$. By Lemma 1 we can then find an $r \in (s, t)$ such that

$$x''(r) = \frac{2z}{3m} - \frac{kA}{m}.$$

Since $r \in (t-\delta, t-\delta)$ and $r \neq t$ we know that $r \in S_1$ and hence

$$mx''(r) + kx(r) = 0.$$

From the preceding two equations it follows that

$$|x(r) - A| = \frac{2|z|}{3k} > \epsilon,$$

which is impossible. Thus the assumption that $x''(t) < 0$ leads to a contradiction, and $x''(t) = 0$.

Suppose now that $x(t) \in S_2$. By the continuity of x there is a $\delta > 0$ such that $t - \delta < s < t + \delta$ implies $x(s) > 2A/3$. Obviously, no such s can belong to S_3. Suppose one belongs to S_1. Then, since $mx''(s) + kx(s) = 0$, we have $x''(s) < -2kA/3$. By Lemma 1, there is a point $r \in (s, t)$ such that

$$x''(r) = -kA/2.$$

r cannot belong to S_2, since $x''(r) \neq 0$. It therefore belongs to S_1,

$$mx''(r) + kx(r) = 0.$$

and hence $x(r) = A/2$. This, however, is impossible, since $r \in (t - \delta, t + \delta)$. The assumption that some $s \in (t - \delta, t + \delta)$ belongs to S_1 thus leads to a contradiction, and all $s \in (t - \delta, t + \delta)$ must belong to S_2.

An almost identical argument, differing only in signs, shows that if $t \in S_3$ then there is a $\delta > 0$ such that $(t - \delta, t + \delta) \subset S_3$. The hypotheses of Lemma 2 are therefore satisfied and we conclude that one of the sets S_1, S_2 or S_3 contains all of $\mathbf{R}$.

Finally, we need to prove the two lemmata. For Lemma 1, consider the function g defined by

$$g(t) = \pm[f(t) - yt]$$

where we take the positive sign in the case $f'(a) < y < f'(b)$ and the negative sign in the case $f'(b) < y < f'(a)$. Then g is differentiable in $[a, b]$ and

$$g'(a) < 0 < g'(b).$$

Since g is differentiable and hence continuous, it must have a minimum. This minumum cannot be at a or b because the derivatives have the wrong signs. The minimum is therefore at some point $t \in (a, b)$. At an interior maximum the derivative, if it exists, is zero. In our case we know it exists, so $g'(t) = 0$ and hence $f'(0) = y$.

For the proof of Lemma 2, consider the function f which is equal to j in S_j. The hypotheses of Lemma 2 imply that f is defined everywhere and is continuous. Suppose there are points a and b such that $a \in S_j$ and $b \in S_k$ where $j \neq k$. Let z be any number between j and k which is no an integer. By the intermediate value theorem there is a $t \in (a, b)$ such that $f(t) = z$. But f takes only integer values, so out assumption that there are points a and b such that $a \in S_j$ and $b \in S_k$ where $j \neq k$ must have been false. Therefore all points belong to the same set S_j.

(b) Prove that all solutions of the differential equation

$$mx''(t) + kx(t) = 0$$

are of the form

$$x(t) = x(0)\cos(\omega t) + \frac{x'(0)}{\omega}\sin(\omega t)$$

where

$$\omega = \sqrt{k/m}.$$

Hint: Let x be an arbitrary solution of the equation and consider the quantities

$$\xi(t) = x(t)\cos(\omega t) - \frac{x'(t)}{\omega}\sin(\omega t)$$

and

$$\eta(t) = x(t)\omega\sin(\omega t) + x'(t)\cos(\omega t).$$

What can be said about their derivatives?
Solution: Differentiating and doing a bit of algebra,

$$\xi'(t) = (km)^{-1/2}\sin(\omega t)(mx''(t) + kx(t))$$

$$\eta'(t) = m^-1\cos(\omega t)(mx''(t) + kx(t))$$

For solutions of the differential equation, both derivatives are zero, so ξ and η are constant. Thus

$$x(t)\cos(\omega t) - \frac{x'(t)}{\omega}\sin(\omega t) = x(0)$$

and

$$x(t)\omega \sin(\omega t) + x'(t)\cos(\omega t) = x'(0)$$

Multiplying the first equation by $\cos(\omega t)$ and the second by $\omega^{-1}\sin(\omega t)$ and adding leads to

$$x(t) = x(0)\cos(\omega t) + \frac{x'(0)}{\omega}\sin(\omega t)$$

(c) Prove that the function

$$x(t) = \begin{cases} -A & \text{if } -\infty < t < -\frac{\pi}{2\omega}\ , \\ A\sin(\omega t) & \text{if } -\frac{\pi}{2\omega} < t < \frac{\pi}{2\omega}\ , \\ A & \text{if } \frac{\pi}{2\omega} < t < \infty\ , \end{cases}$$

which was shown in lecture to be differentiable, is not twice differentiable.

Solution: If it were twice differentiable then, by part (a), it would either be constant, which it clearly is not, or a solution to $mx''(t) + kx(t) = 0$. We calculated all solutions to the latter equation in part (b), and x is not one of them.