MA 216
Assignment 4
Due 5 December 2007

1. Prove, in the 2×2 case, the assertion made in lecture that if W is an invertible differentiable matrix valued function then

$$\frac{d}{dt}\det(W(t)) = \det(W(t))\,\mathrm{tr}(W'(t)W^{-1})$$

2. As shown in lecture, if we have a single nontrivial solution x_1 to the homogeneous equation

$$p(t)x''(t) + q(t)x'(t) + r(t)x(t) = 0$$

then we can, by the method of Wronski, find a second solution x_2, such that x_1 and x_2 form a basis for the set of solutions. Show that if x_1 and x_2 are such a basis then

$$W(t) = \begin{pmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{pmatrix} \begin{pmatrix} x_1(0) & x_2(0) \\ x_1'(0) & x_2'(0) \end{pmatrix}^{-1}$$

is the unique solution to the matrix initial value problem

$$W'(t) = A(t)W(t) \qquad W(0) = I$$

where

$$A(t) = \begin{pmatrix} 0 & 1 \\ -r(t)/p(t) & -q(t)/p(t) \end{pmatrix}.$$

3. Assuming the result of the previous exercise, find a solution to the *inhomogeneous* equation

$$p(t)x''(t) + q(t)x'(t) + r(t)x(t) = \varphi(t)$$

in terms of a pair x_1, x_2 of solutions to the homogeneous equation

$$p(t)x''(t) + q(t)x'(t) + r(t)x(t) = 0.$$

Hint: The equivalent first order system is

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = A(t) \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} 0 \\ \varphi(t)/p(t) \end{pmatrix}.$$

Use the general formula

$$\vec{x}(t) = W(t)\vec{x}(0) + \int_0^t W(t)W(s)^{-1}\vec{f}(s)\, ds,$$

for the solution to a first order inhomogeneous system

$$\vec{x}'(t) = A(t)\vec{x}(t) + \vec{f}(t).$$

4. The Chebyshev equation of order n is

$$(1 - t^2)x''(t) - tx'(t) + n^2 x(t) = 0.$$

Special solutions are

$$x_1(t) = 1$$

when $n = 0$,

$$x_1(t) = t$$

when $n = 1$ and

$$x_1(t) = 2t^2 - 1$$

when $n = 2$. In each of these three cases, find the general solution. *Hint:* You will need to evaluate integrals of the form $\int R(t, \sqrt{1 - t^2})\, dt$, where R is a rational function of two variables. Such integrals can always be evaluated in terms of elementary functions by means of the rationalising substution $t = 2u/(1 + u^2)$ and partial fractions.