MA 216
Assignment 1
Due 23 October 2007

1. For each of the following, say whether it is a scalar equation or a system, what its order is, whether it is linear or nonlinear, and, if linear, whether it is homogeneous or inhomogeneous.

   (a) Bessel's equation:

   $$t^2 x''(t) + t x'(t) + (t^2 - \nu^2) x(t) = 0,$$

   (b) Jacobi's modular equation:

   $$-3k''(t)^2 + 2k'(t)k'''(t) + \left(\frac{1 + k(t)^2}{k(t) - k(t)^3}\right)^2 k'(t)^4 - \left(\frac{1 + t^2}{t - t^3}\right)^2 k'(t)^2 = 0,$$

   (c) The equation of motion of a pendulum:

   $$l\theta''(t) + g\sin(\theta(t)) = 0$$

   (d) The Lorenz equations

   $$x'(t) = \sigma \cdot (y(t) - x(t))$$

   $$y'(t) = x(t)(\rho - z(t)) - y(t)$$
   $$z'(t) = x(t)y(t) - \beta z(t),$$

   (e) The mortgage repayment equation:

   $$x'(t) = rx(t) - A,$$

(f) The circular motion equations:

$$x'(t) = -y(t)$$

$$y'(t) = x(t).$$

2. (a) Prove that

$$E(t) = x(t)^2 + y(t)^2$$

is an invariant of the system

$$x'(t) = -y(t)y'(t) = x(t).$$

and use this fact to obtain bounds on $x(t)$ and $y(t)$ in terms of $x(0)$ and $y(0)$.

(b) Prove that

$$E(t) = x(t)y(t)$$

is an invariant of the system

$$x'(t) = -x(t)$$

$$y'(t) = y(t).$$

This fact does not provide any useful bounds on $x(t)$ and $y(t)$. What makes this example different from the previous one?

3. (a) The Euler equations for a rigid body

$$I_1\Omega_1'(t) = (I_2 - I_3)\Omega_2(t)\Omega_3(t)$$

$$I_2\Omega_2'(t) = (I_3 - I_1)\Omega_3(t)\Omega_1(t)$$

$$I_3\Omega_3'(t) = (I_1 - I_2)\Omega_1(t)\Omega_2(t)$$

which were shown in lecture to possess an invariant

$$E(t) = \frac{1}{2}I_1\Omega_1(t)^2 + \frac{1}{2}I_2\Omega_2(t)^2 + \frac{1}{2}I_3\Omega_3(t)^2$$

have another invariant, which is also quadratic in the variables Ω_1, Ω_2 and Ω_3. Find it.

(b) The equations are unchanged by cyclically permuting the indices 1, 2 and 3. The transformation

$$\tilde{\Omega}_1(t) = \Omega_2(t)$$
$$\tilde{\Omega}_2(t) = \Omega_3(t)$$
$$\tilde{\Omega}_3(t) = \Omega_1(t)$$

is not, however, a symmetry of the system. Why not?

4. (a) Prove that any twice differentiable solution of the equation

$$\frac{1}{2}mx'(t)^2 + \frac{1}{2}kx(t)^2 = E$$

is either constant $\pm\sqrt{E/2k}$ or is a solution of the differential equation

$$mx''(t) + kx(t) = 0.$$

(b) Prove that all solutions of the differential equation

$$mx''(t) + kx(t) = 0$$

are of the form

$$x(t) = x(0)\cos(\omega t) + \frac{x'(0)}{\omega}\sin(\omega t)$$

where

$$\omega = \sqrt{k/m}.$$

Hint: Let x be an arbitrary solution of the equation and consider the quantities

$$\xi(t) = x(t)\cos(\omega t) - \frac{x'(t)}{\omega}\sin(\omega t)$$

and

$$\eta(t) = x(t)\omega\sin(\omega t) + x'(t)\cos(\omega t).$$

What can be said about their derivatives?

(c) Prove that the function

$$x(t) = \begin{cases} -A & \text{if } -\infty < t < -\frac{\pi}{2\omega}\ , \\ A\sin(\omega t) & \text{if } -\frac{\pi}{2\omega} < t < \frac{\pi}{2\omega}\ , \\ A & \text{if } \frac{\pi}{2\omega} < t < \infty\ , \end{cases}$$

which was shown in lecture to be differentiable, is not twice differentiable.