UNIVERSITY OF DUBLIN

XMA2161

TRINITY COLLEGE

FACULTY OF SCIENCE

SCHOOL OF MATHEMATICS

SF Maths
SF TP

Trinity Term 2007

COURSE 216

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ATTEMPT FOUR QUESTIONS

Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

1. *(25 points)* For each of the following equations, state whether *all* solutions are bounded for $t > 0$ and whether *all* solutions satisfy

$$\lim_{t \to +\infty} x(t) = 0.$$

While you needn't solve the equations explicitly, you should indicate how your answers were obtained.

(a) *(5 points)*

$$x''(t) + 3x'(t) + 2x(t) = 0$$

(b) *(5 points)*

$$x''(t) + 2x'(t) - 3x(t) = 0$$

(c) *(5 points)*

$$x''(t) - 2x'(t) + 2x(t) = 0$$

(d) *(5 points)*

$$x''(t) + 2x'(t) + 2x(t) = 0$$

(e) *(5 points)*

$$x''(t) - 9x(t) = 0$$

2. *(25 points)*

   (a) *(10 points)* Give an example of matrices A and B such that

   $$\exp(A+B) \neq \exp(A)\exp(B)$$

   Give full details, *i.e.* give $\exp(A)$, $\exp(B)$ and $\exp(A+B)$ as well as A and B. You don't need to show how $\exp(A)$, $\exp(B)$ and $\exp(A+B)$ were calculated, however.

   (b) *(15 points)* Given that

   $$A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix},$$

   compute $\exp(tA)$.

3. *(25 points)* Given that

   $$x_1(t) = t$$

   is a solution of the equation

   $$(t^2+2)x''(t) - 2tx'(t) + 2x(t) = 0,$$

   find the general solution in terms of $x(0)$ and $x'(0)$.

4. *(25 points)* Laguerre's equation is

$$tx''(t) + (1-t)x'(t) + \nu x(t) = 0.$$

For which values of ν does this equation have polynomial solutions? Give the polynomial solutions corresponding to at least three distinct values of ν.

5. *(25 points)*

(a) *(3 points)* Prove that $(0,0)$ is an equilibrium of the autonomous system

$$x' = -15x^3 - 24x^2y - 15xy^2 - 4x - 5y,$$

$$y' = -15x^2y - 24xy^2 - 15y^3 + 5x + 4y.$$

(b) *(8 points)* Prove that

$$V(x,y) = 5x^2 + 8xy + 5y^2$$

is a Lyapunov function, but not a strict Lyapunov function.

(c) *(4 points)* What conclusions can you draw about stability or asymptotic stability from the existence of a Lyapunov function?

(d) *(10 points)* Prove that *all* solutions are bounded.

6. *(25 points)*

   (a) *(10 points)* Find a basis for the space of solutions to the linear homogeneous differential equation

   $$x''''(t) - x(t) = 0.$$

   (b) *(15 points)* Find all solutions of the linear inhomogenous differential equation

   $$x''''(t) - x(t) = t.$$