UNIVERSITY OF DUBLIN

XMA2161

TRINITY COLLEGE

Faculty of Science

school of mathematics

SF Maths
SF TP

Trinity Term 2007

Course 216

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ATTEMPT FOUR QUESTIONS

Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

1. *(25 points)* For each of the following equations, state whether *all* soltutions are bounded for $t > 0$ and whether *all* solutions satisfy

$$\lim_{t\to+\infty} x(t) = 0.$$

While you needn't solve the equations explicitly, you should indicate how your answers were obtained.

(a) *(5 points)*

$$x''(t) + 2x'(t) + 3x(t) = 0$$

(b) *(5 points)*

$$x''(t) - 2x'(t) + 3x(t) = 0$$

(c) *(5 points)*

$$x''(t) + 4x(t) = 0$$

(d) *(5 points)*

$$x''(t) - 4x(t) = 0$$

(e) *(5 points)*

$$x'''(t) + x''(t) = 0$$

2. *(25 points)*

   (a) *(5 points)* How is the matrix exponential defined? *Note: I'm not asking how it's computed, I'm asking how it's defined.*

   (b) *(5 points)* What useful property of the ordinary exponential function does the matrix exponential fail to satisfy?

   (c) *(15 points)* Given that

$$\begin{pmatrix} -43 & -63 \\ 30 & 44 \end{pmatrix} \begin{pmatrix} -7 & 3 \\ 5 & -2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix},$$

   express the general solution to the system

$$x'(t) = -43x(t) - 63y(t)$$

$$y'(t) = 30x(t) + 44y(t)$$

   in terms of the initial values $x(0)$ and $y(0)$.

3. *(25 points)*

   (a) *(5 points)* Prove that

$$A = x^2 + y^2$$

   and

$$B = x^2 + z^2$$

   are invariants of the nonlinear system

$$x'(t) = y(t)z(t)$$

$$y'(t) = -x(t)z(t)$$

$$z'(t) = -x(t)y(t)$$

   (b) *(10 points)* Prove that all solutions of the system are bounded.

   (c) *(10 points)* Prove that $(0, 0, 0)$ is a stable equilibrium.

4. *(25 points)* Given that

$$x_1(t) = t^2 + 1$$

is a solution of the equation

$$(t^2 - 2t - 1)x''(t) + (-2t + 2)x'(t) + 2x(t) = 0,$$

find *all* solutions.

5. *(25 points)*

   (a) *(16 points)* Define the following terms:

      i. Autonomous System
      ii. Equilibrium
      iii. Stable
      iv. Asymptotically stable
      v. Strictly stable
      vi. Linearisation
      vii. Lyapunov function
      viii. Strict Lyapunov function

   (b) *(9 points)* Prove that

   $$V(x, y) = x^2 + y^2$$

   is a strict Lyapunov function for the equilibrium $(0, 0)$ of the system

   $$x'(t) = -x(t) + y(t)$$

   $$y'(t) = -x(t) - y(t)$$

6. *(25 points)* Chebyshev's equation is

$$(1 - t^2)x''(t) - t'x(t) + \nu^2 x(t) = 0.$$

For which values of ν does this equation have polynomial solutions? What are these solutions?