MA 419 Assignment 6

Due 18 April 2007

Solutions

1. The Camassa-Holm equation

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0$$

is in many ways very similar to the Korteweg de Vries equation. It has soliton solutions and infinitely many essentially different conservation laws. Prove the first two of these conservation laws, that the quantities

$$A = \int_{-\infty}^{+\infty} u \, dx$$

and

$$B = \int_{-\infty}^{+\infty} (u^2 + u_x^2) \, dx$$

are constant for reasonable solutions. You may take the word "reasonable" to mean Schwartz class in the x variable, though much weaker hypotheses would suffice.

Hint: This is best done by writing the conservation law in divergence form, *i.e.* $\partial P/\partial t + \partial Q/\partial x = 0$, as explained in lecture. P is your integrand. You need to find an appropriate Q.

Solution: In the first case we take

$$P = u \qquad Q = -u_{tx} + \frac{3}{2}u^2 - \frac{1}{2}u_x^2 - uu_{xx}$$

while in the second case we take

$$P = u^2 + u_x^2 \qquad Q = u^3 + u^2 u_{xx} - 2uu_{tx}$$

In either case,

$$P_t + Q_x = 0.$$

From here there are two ways to proceed. Either we apply the Divergence Theorem to the rectangle

$$t_1 \le t \le t_2 \qquad x_1 \le x \le x_2$$

to obtain

$$\int_{x_1}^{x_2} P|_{t=t_2} \, dx - \int_{x_1}^{x_2} P|_{t=t_2} \, dx = \int_{t_1}^{t_2} Q|_{x=x_2} \, dt - \int_{t_1}^{t_2} Q|_{x=x_1} \, dt.$$

The right hand side tends to zero, because of the assumption that u is of Schwartz class in x, so

$$\int_{-\infty}^{+\infty} P|_{t=t_2}\,dx = \int_{-\infty}^{+\infty} P|_{t=t_1}\,dx$$

for all t_1 and t_2. Alternatively, we can differentiate under the integral sign,

$$\frac{d}{dt}\int_{-\infty}^{+\infty} P\,dx = \int_{-\infty}^{+\infty} P_t\,dx = -\int_{-\infty}^{+\infty} Q_x\,dx.$$

This is justified by the convergence of the resulting integral, which again follows from the assumption that u is of Schwartz class in x. By the Fundamental Theorem of the Calculus,

$$\int_{-\infty}^{+\infty} Q_x\,dx = \lim_{\substack{x_1\to-\infty\\ x_2\to+\infty}} \left(Q(t,x_2) - Q(t,x_1)\right).$$

The right hand side is zero, again because of the assumption that u is of Schwartz class in x.

2. Prove the assertion made in lecture, that

$$\tilde{u}(x_1,\ldots,x_n) = r^{2-n} u\left(\frac{x_1}{r^2},\ldots,\frac{x_n}{r^2}\right)$$

is harmonic if and only if u is. Here, as in lecture, $r^2 = x_1^2 + \cdots + x_n^2$. *Solution:* This is simply a calculation, though an exceptionally long one. To make it slightly less ugly, set

$$\tilde{x}_i = \frac{x_i}{r^2}.$$

Then

$$\begin{aligned}\frac{\partial \tilde{u}}{\partial x_j}(x_1,\ldots,x_n) &= r^{-n}\frac{\partial u}{\partial \tilde{x}_j}(\tilde{x}_1,\ldots,\tilde{x}_n)\\ &\quad - 2\sum_{k=1}^{n} r^{-n-2} x_j x_k \frac{\partial u}{\partial \tilde{x}_k}(\tilde{x}_1,\ldots,\tilde{x}_n)\\ &\quad - (n-2)\, r^{-n} x_j u(\tilde{x}_1,\ldots,\tilde{x}_n)\end{aligned}$$

Differentiating each summand on the right,

$$\begin{aligned}\frac{\partial}{\partial x_j}\left(r^{-n}\frac{\partial u}{\partial \tilde{x}_j}(\tilde{x}_1,\ldots,\tilde{x}_n)\right) &= r^{-n-2}\frac{\partial^2 u}{\partial \tilde{x}_j^2}(\tilde{x}_1,\ldots,\tilde{x}_n)\\ &\quad - 2\sum_{k=1}^{n} r^{-n-4} x_j x_k \frac{\partial^2 u}{\partial \tilde{x}_j \partial \tilde{x}_k}(\tilde{x}_1,\ldots,\tilde{x}_n)\\ &\quad - n r^{-n-2} x_j \frac{\partial u}{\partial \tilde{x}_j}(\tilde{x}_1,\ldots,\tilde{x}_n),\end{aligned}$$

$$
\begin{aligned}
\frac{\partial}{\partial x_j}\left(\sum_{k=1}^{n} r^{-n-2} x_j x_k \frac{\partial u}{\partial \tilde{x}_k}(\tilde{x}_1,\ldots,\tilde{x}_n)\right) = \\
&\sum_{k=1}^{n} r^{-n-4} x_j x_k \frac{\partial^2 u}{\partial \tilde{x}_j \partial \tilde{x}_k}(\tilde{x}_1,\ldots,\tilde{x}_n)\\
&- 2\sum_{k=1}^{n}\sum_{l=1}^{n} r^{-n-6} x_j^2 x_k x_l \frac{\partial^2 u}{\partial \tilde{x}_k \partial \tilde{x}_l}(\tilde{x}_1,\ldots,\tilde{x}_n)\\
&+ \sum_{k=1}^{n} r^{-n-2} x_k \frac{\partial u}{\partial \tilde{x}_k}(\tilde{x}_1,\ldots,\tilde{x}_n)\\
&+ r^{-n-2} x_j \frac{\partial u}{\partial \tilde{x}_j}(\tilde{x}_1,\ldots,\tilde{x}_n)\\
&- (n+2)\, r^{-n-2} x_j^2 x_k \frac{\partial u}{\partial \tilde{x}_k}(\tilde{x}_1,\ldots,\tilde{x}_n)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial x_j}\left(r^{-n-2} x_j u(\tilde{x}_1,\ldots,\tilde{x}_n)\right) = {} & r^{-n-4} x_j \frac{\partial u}{\partial \tilde{x}_j}(\tilde{x}_1,\ldots,\tilde{x}_n)\\
&- 2\sum_{k=1}^{n} r^{-n-6} x_j^2 x_k \frac{\partial u}{\partial \tilde{x}_k}(\tilde{x}_1,\ldots,\tilde{x}_n)\\
&- n\, r^{-n-2} x_j^2 u(\tilde{x}_1,\ldots,\tilde{x}_n)\\
&+ r^{-n-4} u(\tilde{x}_1,\ldots,\tilde{x}_n).
\end{aligned}
$$

Combining these,

$$
\begin{aligned}
\frac{\partial^2 \tilde{u}}{\partial x_j^2}(x_1,\cdots,x_n) = {} & r^{-n-2} \frac{\partial^2 u}{\partial \tilde{x}_j^2}(\tilde{x}_1,\ldots,\tilde{x}_n)\\
&- 4\sum_{k=1}^{n} r^{-n-4} x_j x_k \frac{\partial^2 u}{\partial \tilde{x}_j \partial \tilde{x}_k}(\tilde{x}_1,\ldots,\tilde{x}_n)\\
&+ 4\sum_{k=1}^{n}\sum_{l=1}^{n} r^{-n-6} x_j^2 x_k x_l \frac{\partial^2 u}{\partial \tilde{x}_k \partial \tilde{x}_l}(\tilde{x}_1,\ldots,\tilde{x}_n)\\
&- 2n r^{-n-4} x_j \frac{\partial u}{\partial \tilde{x}_j}(\tilde{x}_1,\ldots,\tilde{x}_n)\\
&- 2\sum_{k=1}^{n} r^{-n-2} x_k \frac{\partial u}{\partial \tilde{x}_k}(\tilde{x}_1,\ldots,\tilde{x}_n)\\
&+ 4n\, r^{-n-2} x_j^2 x_k \frac{\partial u}{\partial \tilde{x}_k}(\tilde{x}_1,\ldots,\tilde{x}_n)\\
&+ n(n-2)\, r^{-n-2} x_j^2 u(\tilde{x}_1,\ldots,\tilde{x}_n)\\
&- (n-2) r^{-n-4} u(\tilde{x}_1,\ldots,\tilde{x}_n).
\end{aligned}
$$

Summing over j gives

$$
\sum_{j=1}^{n} \frac{\partial^2 \tilde{u}}{\partial x_j^2} = r^{-n-2} \sum_{j=1}^{n} \frac{\partial^2 u}{\partial \tilde{x}_j^2}
$$

so

$$\sum_{j=1}^{n} \frac{\partial^2 \tilde{u}}{\partial x_j^2} = 0$$

if and only if

$$\sum_{j=1}^{n} \frac{\partial^2 u}{\partial \tilde{x}_j^2} = 0$$

3. Suppose that f is continuous and bounded on $\mathbf{R}^2$ and that u is defined in the upper halfspace

$$H = \{(x, y, z) \colon z \geq 0\}$$

by

$$u(x, y, 0) = f(x, y)$$

and by the Poisson formula

$$u(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} K(x, y, z, \xi, \eta) f(\xi, \eta) \, d\xi \, d\eta$$

for $z > 0$, where

$$K(x, y, z, \xi, \eta) = \frac{z}{((x-\xi)^2 + (y-\eta)^2 + z^2)^{3/2}}.$$

Prove that u is harmonic in the interior of H.

Hint: The most straightforward way to do this is to differentiate under the integral sign. In principle you should check that this is justified, *i.e.* that the resulting integrals converge. There are less painful ways, but these require an understanding of where the given K comes from. It is probably easier to apply brute force than to try to be clever.
Solution: Differentiating,

$$\begin{aligned}
\frac{\partial K}{\partial x} &= -3 \frac{(x-\xi)z}{((x-\xi)^2 + (y-\eta)^2 + z^2)^{5/2}}, \\
\frac{\partial^2 K}{\partial x^2} &= 15 \frac{(x-\xi)^2 z}{((x-\xi)^2 + (y-\eta)^2 + z^2)^{7/2}} \\
&\quad - 3 \frac{z}{((x-\xi)^2 + (y-\eta)^2 + z^2)^{5/2}}, \\
\frac{\partial K}{\partial y} &= -3 \frac{(y-\eta)z}{((x-\xi)^2 + (y-\eta)^2 + z^2)^{5/2}}
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 K}{\partial y^2} &= 15\frac{(y-\eta)^2 z}{((x-\xi)^2+(y-\eta)^2+z^2)^{7/2}} \\
&\quad - 3\frac{z}{((x-\xi)^2+(y-\eta)^2+z^2)^{5/2}},
\end{aligned}
$$

$$
\frac{\partial K}{\partial z} = \frac{1}{((x-\xi)^2+(y-\eta)^2+z^2)^{3/2}} - 3\frac{z^2}{((x-\xi)^2+(y-\eta)^2+z^2)^{5/2}}
$$

and

$$
\begin{aligned}
\frac{\partial^2 K}{\partial z^2} &= 15\frac{z^3}{((x-\xi)^2+(y-\eta)^2+z^2)^{7/2}} \\
&\quad - 9\frac{z}{((x-\xi)^2+(y-\eta)^2+z^2)^{5/2}}.
\end{aligned}
$$

Summing

$$
\frac{\partial^2 K}{\partial x^2} + \frac{\partial^2 K}{\partial y^2} + \frac{\partial^2 K}{\partial z^2} = 0.
$$

Differentiating under the integral sign gives

$$
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.
$$

In order to justify this we should check that the integrals

$$
\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\partial^2 K}{\partial x^2}(x,y,z,\xi,\eta) f(\xi,\eta)\, d\xi\, d\eta,
$$

$$
\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\partial^2 K}{\partial y^2}(x,y,z,\xi,\eta) f(\xi,\eta)\, d\xi\, d\eta
$$

and

$$
\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\partial^2 K}{\partial z^2}(x,y,z,\xi,\eta) f(\xi,\eta)\, d\xi\, d\eta,
$$

converge. This follows, for example, from the fact that all of the integrands are bounded by a a multiple of $(1+\xi^2+\eta^2)^{-2}$, and the latter is certainly integrable.

4. With u and f as in the previous problem, prove that u is bounded in H.

   Hint: u is given by a convolution. Apply Young's Inequality.
   Solution:

$$
u = \frac{1}{2\pi} K(x,y,z,\cdot,\cdot) \star f
$$

so Young's inequality gives

$$
\|u\|_{L^r(\mathbf{R}^2)} \leq \frac{1}{2\pi}\|K(x,y,z,\cdot,\cdot)\|_{L^p(\mathbf{R}^2)}\|_{L^q(\mathbf{R}^2)}
$$

if

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1.$$

For this problem we need the case $p = 1$, $q = r = \infty$. The finiteness of

$$\|K(x, y, z, \cdot, \cdot)\|_{L^1(\mathbf{R}^2)} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{z}{((x-\xi)^2 + (y-\eta)^2 + z^2)^{3/2}} \, d\xi \, d\eta$$

is fairly easy to prove. The numerical value is not needed here, but will be calculated in the solution to the next problem, giving the sharp estimate

$$\|u\|_{L^\infty(\mathbf{R}^2)} \leq \|f\|_{L^\infty(\mathbf{R}^2)}.$$

5. With u and f as in the previous two problems, prove that u is continuous on H.

   Hint: I gave three different arguments for the Poisson Formula for the unit disc in the plane. Each of these has analogues for the upper half space in $\mathbf{R}^3$, but some are easier than others. What I would recommend is to make the substitution

$$s = \frac{\xi - x}{z} \qquad t = \frac{\eta - y}{z}$$

   in the integral, and then to use Lebesgue Dominated Convergence.
   Solution: Continuity in the interior of the upper half space follows immediately from differentiability. The hard part is therefore continuity at the boundary. In other words, we need to establish that

$$\lim_{(x,y,z)\to(X,Y,0)} u(x, y, z) = f(X, Y).$$

   for any $(X, Y) \in \mathbf{R}^2$.

   Making the suggested substitution,

$$u(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{(1 + s^2 + t^2)^{3/2}} f(x + sz, y + tz) \, ds \, dt.$$

   By the Lebesgue Dominated Convergence Theorem,

$$\lim_{(x,y,z)\to(X,Y,0)} u(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{f(X, Y) \, ds \, dt}{(1 + s^2 + t^2)^{3/2}} = F(X, Y).$$

   The integral was evaluated as follows. Switching to polar coordinates,

$$\begin{aligned} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{ds \, dt}{(1 + s^2 + t^2)^{3/2}} &= \int_0^{2\pi} \int_0^{\infty} \frac{r \, dr \, d\theta}{(1 + r^2)^{3/2}} \\ &= \int_0^{2\pi} \int_0^{\infty} \frac{2\pi r \, dr}{(1 + r^2)^{3/2}} \\ &= \int_1^{\infty} \frac{\pi \, dv}{v^{3/2}} = 2\pi. \end{aligned}$$