MA 419 Assignment 5

Due 28 February 2007

*Solutions*

1. Prove that
$$u = \begin{cases} \frac{1}{2c} & \text{if } -ct \le x \le ct \\ 0 & \text{otherwise} \end{cases}$$
is a distribution solution to the inhomogeneous wave equation
$$u_{tt} - c^2 u_{xx} = \delta.$$
*Hint:* After consulting the definitions, you should see that this is a statement about the value of a certain integral. In order to prove that statement it is convenient to switch to characteristic coordinates.
*Solution:* The definitions of derivatives of distributions and of the the delta distribution show that the distribution equation $u_{tt} - c^2 u_{xx} = \delta$ is really just the statement that
$$\int_{\mathbf{R}^2} (\psi_{tt} - c^2 \psi_{xx}) u \, dx \, dt = \psi(0,0)$$
for all Schwartz class functions $\psi$. Changing variables to
$$\xi = x + ct \qquad \eta = x - ct,$$
one sees that this statement is equivalent to
$$\int_{\mathbf{R}^2} 2cu \left( \frac{\xi - \eta}{2c}, \frac{\xi + \eta}{2} \right) \psi_{\xi,\eta} \, d\xi \, d\eta = \psi(0,0)$$
or, using the definition of $u$,
$$\int_{-\infty}^{0} \int_{0}^{\infty} \psi_{\xi\eta} \, d\xi \, d\eta = \psi(0,0).$$
This statement follows from the Fundamental Theorem of the Calculus, the boundary terms at infinity vanishing because of the assumption that $\psi$ is in Schwarz class.

2. Solve the homogeneous Laplace equation
$$u_{xx} + u_{yy} = 0$$
in the upper half plane with Dirichlet boundary conditions
$$u(x,0) = \frac{x^2 - 1}{x^2 + 1}.$$
*Solution:* Using the Poisson Formula,
$$u(x,y) = \frac{1}{\pi} \int_{-\infty}^{+\infty} F(x,y,z) \, dz.$$

where

$$F(x,y,z) = \frac{(z^2-1)y}{(z^2+1)[(x-z)^2+y^2]}.$$

$F$ is a rational function of $z$ with simple poles at the points $\pm i$ and $x \pm iy$ and a zero at infinity. Although it is possible to do the partial fractions calculation with real polynomials using Linear Algebra it is easier to use a bit of Complex Analysis. Since $F$ has simple zeroes poles and a zero at infinity

$$F(x,y,z) = \sum_{F(w)=\infty} \frac{\mathrm{Res}_{z=w} F(z)}{z-w}.$$

Then

$$\begin{aligned}
u(x,y) &= \frac{1}{\pi}\int_{-\infty}^{+\infty} F(x,y,z)\,dz \\
&= \frac{1}{\pi}\lim_{R\to\infty}\int_{-R}^{R} F(x,y,z)\,dz \\
&= \sum_{F(x,y,w)=\infty} \operatorname*{Res}_{z=w} F(x,y,z) \lim_{R\to\infty} \frac{1}{\pi}\int_{-R}^{R}\frac{dz}{z-w} \\
&= \sum_{F(x,y,w)=\infty} \operatorname*{Res}_{z=w} F(x,y,z) \lim_{R\to\infty} \frac{\log\left(\frac{R-w}{-R-w}\right)}{\pi} \\
&= \sum_{F(x,y,w)=\infty} \operatorname*{Res}_{z=w} F(x,y,z) \lim_{R\to\infty} \frac{\log\left(-\frac{1-w/R}{1+w/R}\right)}{\pi} \\
&= \sum_{F(x,y,w)=\infty} \pm 2i \operatorname*{Res}_{z=w} F(x,y,z)
\end{aligned}$$

where the $\pm$ is the sign of the imaginary part of $w$. Calculating the residues using the relation

$$\operatorname*{Res}_{z=w} F(x,y,z) = \lim_{z\to w}(z-w)F(x,y,z),$$

we get

$$\operatorname*{Res}_{z=i} F(x,y,z) = \frac{i}{(x-i)^2+y^2},$$

$$\operatorname*{Res}_{z=-i} F(x,y,z) = \frac{-i}{(x+i)^2+y^2},$$

$$\operatorname*{Res}_{z=x+iy} F(x,y,z) = \frac{-i[(x+iy)^2-1]}{2[(x+iy)^2-1]y}$$

and

$$\operatorname*{Res}_{z=x-iy} F(x,y,z) = \frac{i[(x-iy)^2-1]}{2[(x-iy)^2-1]y}.$$

Substituting, and doing a bit of algebra,

$$u(x,y) = \frac{x^2+y^2-1}{x^2+y^2+2y+1}.$$

3. Find an integral formula for the solution of the *inhomogeneous* Laplace equation

$$u_{xx} + u_{yy} = f$$

in the half plane with Dirichlet boundary conditions

$$u(x,0) = \varphi(x).$$

*Hint:* The equation is linear, so it suffices to solve the problems

$$u_{xx} + u_{yy} = 0 \quad u(x,0) = \varphi(x)$$

and

$$u_{xx} + u_{yy} = f \quad u(x,0) = 0$$

and then add the solutions. The former problem was solved in lecture. The latter problem was not. I solved $u_{xx} + u_{yy} = f$ in the whole plane without boundary conditions. To get the solution in the half plane with boundary conditions you can use the Method of Reflection.
*Solution:* As indicated in the hint, if we can find functions $v$ and $w$ such that

$$v_{xx} + v_{yy} = 0 \quad v(x,0) = \varphi(x)$$

and

$$w_{xx} + w_{yy} = f \quad w(x,0) = 0$$

then

$$u = v + w$$

is the solution to our problem. The solution to the first problem, given in lecture, is

$$v(x,y) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{\varphi(z)\,dz}{(z-x)^2 + y^2}\,dz.$$

To find the solution to the second, we use the Method of Reflection. We extend $f$ to the whole plane,

$$\tilde{f}(x,y) = \begin{cases} f(x,y) & \text{if } y > 0 \\ -f(x,-y) & \text{if } y < 0 \end{cases}$$

and solve

$$\tilde{w}_{xx} + \tilde{w}_{yy} = \tilde{f}$$

without boundary conditions in the whole plane,

$$\tilde{w}(x,y) = \frac{1}{4\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \log\left((x-\xi)^2 + (y-\eta)^2\right) \tilde{f}(\xi,\eta)\,d\xi\,d\eta.$$

We then restrict to $y > 0$, splitting the integral,

$$\begin{aligned} w(x,y) = \tilde{w}(x,y) &= \frac{1}{4\pi} \int_{0}^{+\infty} \int_{-\infty}^{+\infty} \log\left((x-\xi)^2 + (y-\eta)^2\right) \tilde{f}(\xi,\eta)\,d\xi\,d\eta \\ &+ \frac{1}{4\pi} \int_{-\infty}^{0} \int_{-\infty}^{+\infty} \log\left((x-\xi)^2 + (y-\eta)^2\right) \tilde{f}(\xi,\eta)\,d\xi\,d\eta \end{aligned}$$

or

$$
\begin{aligned}
w(x,y) &= \frac{1}{4\pi}\int_0^{+\infty}\int_{-\infty}^{+\infty} \log\left((x-\xi)^2+(y-\eta)^2\right) f(\xi,\eta)\,d\xi\,d\eta \\
&\quad - \frac{1}{4\pi}\int_{-\infty}^{0}\int_{-\infty}^{+\infty} \log\left((x-\xi)^2+(y-\eta)^2\right) f(\xi,-\eta)\,d\xi\,d\eta \\
&= \frac{1}{4\pi}\int_0^{+\infty}\int_{-\infty}^{+\infty} \log\left((x-\xi)^2+(y-\eta)^2\right) f(\xi,\eta)\,d\xi\,d\eta \\
&\quad - \frac{1}{4\pi}\int_0^{+\infty}\int_{-\infty}^{+\infty} \log\left((x-\xi)^2+(y+\eta)^2\right) f(\xi,\eta)\,d\xi\,d\eta \\
&= \frac{1}{4\pi}\int_0^{+\infty}\int_{-\infty}^{+\infty} \log\left(\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\eta)^2}\right) f(\xi,\eta)\,d\xi\,d\eta
\end{aligned}
$$

The final answer is therefore

$$
\begin{aligned}
u(x,y) &= \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\varphi(z)\,dz}{(z-x)^2+y^2}\,dz \\
&\quad + \frac{1}{4\pi}\int_0^{+\infty}\int_{-\infty}^{+\infty} \log\left(\frac{(x-\xi)^2+(y-\eta)^2}{(x-\xi)^2+(y+\eta)^2}\right) f(\xi,\eta)\,d\xi\,d\eta.
\end{aligned}
$$

4. Prove that there are constants $C_{j,k}$ such that if $u$ is harmonic in the open disc of radius $R$ about the point $(\xi,\eta)$ and

$$|u(x,y)| \le K$$

for all $(x,y)$ there, then

$$\left|\frac{\partial^{j+k}u}{\partial x^j\partial y^k}(\xi,\eta)\right| \le C_{j,k}KR^{-j-k}$$

You don't need to find explicit $C_{j,k}$'s and certainly shouldn't worry about finding the best possible constants. *Hint:* The case $j=k=0$ is just the Maximum Principle. The case $j+k=1$ was done in lecture. The method used there works in the general case as well, only the calculations are messier. Try to avoid calculating more than you actually need to.
*Solution:* Choose a positive $\alpha<1$ By the Poisson Formula, applied to the disc of radius $a=\alpha R$ about the point $(\xi,\eta)$we have

$$u(\xi+r\cos\theta,\eta+r\sin\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K(r,\theta,a,\varphi)u(\xi+a\cos\varphi,\eta+a\sin\varphi)\,d\theta$$

where

$$K(r,\theta,a,\varphi) = \frac{a^2-r^2}{a^2-2ar\cos(\theta-\varphi)+r^2}$$

It is convenient to switch to a mixture of polar and Cartesian coordinates, *i.e.* to set

$$s=r\cos\theta=x-\xi \qquad t=r\sin\theta=y-eta$$

so that

$$u(x,y)=\frac{1}{2\pi}\int_{-\pi^\pi}\kappa(s,t,a,\varphi)u(\xi+a\cos\varphi,\eta+a\sin\varphi)\,d\varphi$$

where

$$\kappa(s,t,a,\varphi) = K(r,\theta,a,\varphi) = \frac{a^2 - s^2 - t^2}{a^2 - 2as\cos\varphi - 2at\sin\varphi + s^2 + t^2}.$$

Since $\xi$ and $\eta$ are fixed there is no distinction between derivatives with respect to $x$ and derivatives with respect to $s$ or between derivatives with respect to $t$ and derivatives with respect to $y$. Differentiation under the integral sign therefore gives

$$\frac{\partial^{j+k}u}{\partial x^j \partial y^k}(x,y) = \frac{1}{2\pi}\int_{-\pi^\pi} \frac{\partial^{j+k}\kappa}{\partial s^j \partial t^k}(s,t,a,\varphi)u(\xi + a\cos\varphi, \eta + a\sin\varphi)\,d\varphi.$$

A simple inductive argument shows that

$$\frac{\partial^{j+k}\kappa}{\partial s^j \partial t^k}(s,t,a,\varphi) = \frac{p_{j,k}(a,x,y,\varphi)}{(a^2 - 2as\cos\varphi - 2at\sin\varphi + s^2 + t^2)^{j+k+1}}$$

where $p_{j,k}$ is a homogeneous polynomial of degree $j+k+2$ in $s$, $t$ and $a$ with coefficients which are continuous functions of $\varphi$. The base case is clear and the inductive step uses the recurrence relations

$$p_{j+1,k} = (a^2 - 2as\cos\varphi - 2at\sin\varphi + s^2 + t^2)\frac{\partial p_{j,k}}{\partial s} + 2(j+k+1)(s - a\cos\varphi)p_{j,k}$$

$$p_{j,k+1} = (a^2 - 2as\cos\varphi - 2at\sin\varphi + s^2 + t^2)\frac{\partial p_{j,k}}{\partial t} + 2(j+k+1)(t - a\sin\varphi)p_{j,k}.$$

Let $c_{j,k,l,m}(\varphi)$ be the coefficient of $s^l t^m a^{j+k-l-m+2}$ in $p_{j,k}$,

$$p_{j,k}(a,x,y,\varphi) = \sum_{l+m\le j+k+2} c_{j,k,l,m}(\varphi)s^l t^m a^{j+k-l-m+2}.$$

Then, setting $s = t = 0$,

$$\frac{\partial^{j+k}u}{\partial x^j \partial y^k}(\xi,\eta) = \frac{a^{-j-k}}{2\pi}\int_{-\pi^\pi} c_{j,k,0,0}(\varphi)u(\xi + a\cos\varphi, \eta + a\sin\varphi)\,d\varphi.$$

and hence

$$\left|\frac{\partial^{j+k}u}{\partial x^j \partial y^k}(\xi,\eta)\right| \le \max|c_{j,k,0,0}a^{-j-k}| \max_{(x-\xi)^2+(y-\eta^2)=a^2} |u(x,y)| \le$$

so the statement of the problem holds with

$$C_{j-k} = \max|c_{j,k,0,0}a^{-j-k}|\alpha^{-j-k}.$$

5. A function $u$ in the plane is said to be of polynomial growth if there are constants $A$, $\rho$ and $N$ such that

$$x^2 + y^2 \ge \rho^2$$

implies

$$|u(x,y)| \le A(x^2 + y^2)^{N/2}.$$

Assuming the result of the previous problem, even if you didn't manage to prove it, prove that every harmonic function of polynomial growth is a polynomial. *Hint:* It follows from Taylor's theorem that a function is a polynomial if and only if all but finitely many of its partial derivatives are identically zero. Apply the result of the preceding problem to discs of large radius about an arbitrary point.
*Solution:* Suppose

$$R > r + \rho$$

and

$$r = \sqrt{\xi^2 + \eta^2}.$$

Then the circle of radius $R$ about $(\xi, \eta)$ is contained between the circles of radius $\rho$ and $R + r$ about the origin. It follows from the definition of polynomial growth that

$$|u(x, y)| \le A(x^2 + y^2)^{N/2} \le A(R + r)^N \le 2^N AR^N$$

on this circle, and hence, by the maximum principle, throughout the disc of radius $R$ about $(\xi, \eta)$. By the result of the preceding exercise,

$$\left| \frac{\partial^{j+k} u}{\partial x^j \partial y^k}(\xi, \eta) \right| \le 2^N AC_{j,k} R^{N-j-k}$$

Letting $R$ tend to infinity we see that

$$\frac{\partial^{j+k} u}{\partial x^j \partial y^k}(\xi, \eta) = 0$$

if $j + k > N$. No assumption was made about the point $(\xi, \eta)$, so the derivatives of order greater than $N$ vanish identically. It follows that the remainder term in the Taylor formula of order $N$ is zero, and hence that $u$ is a polynomial of order $N$.