MA 419 Assignment 4

Due 31 January 2007

*Solutions*

1. Prove that any bounded weak solution of the Wave Equation

$$u_{tt} - c^2 u_{xx} = 0$$

is a solution in the sense of distributions. Is it true that any weak solution is a distribution solution?
*Solution:* After consulting the definition of distribution derivatives we see that we need to prove

$$\int_{\mathbf{R}^2} u(\varphi_{tt} - c^2 \varphi_{xx} = 0$$

for all $\varphi$ in the Schwarz class $\mathcal{S}(\mathbf{R}^2)$. From the definition of weak solutions we know that

$$\int_{\mathbf{R}^2} u(\psi_{tt} - c^2 \psi_{xx} = 0$$

for all compactly supported smooth $\psi$. This holds in particular for

$$\phi = \rho\varphi$$

where
$$\rho(t, x) = \theta(t/R, x/R),$$
$R > 0$ and $\theta$ is a smooth compactly supported function equal to 1 in the unit ball. Then

$$\begin{aligned} u(\varphi_{tt} - c^2 \varphi_{xx}) = {} & u(\psi_{tt} - c^2 \psi_{xx}) - u\rho_t \varphi_t + c^2 u \rho_x \varphi_x - u\varphi(\rho_{tt} - c^2 \rho_{xx}) \\ & + (1 - \rho)u(\varphi_{tt} - c^2 \varphi_{xx}) \end{aligned}$$

We now integrate both sides over $\mathbf{R}^2$. The integral of the first term on the right is then zero, as we just saw. The integral of the second term is bounded by
$$\|u\|_{L^\infty(\mathbf{R}^2)} \|\rho_t\|_{L^\infty(\mathbf{R}^2)} \|\varphi_t\|_{L^1(\mathbf{R}^2)}.$$
Note that
$$\|\rho_t\|_{L^\infty(\mathbf{R}^2)} = \frac{1}{R} \|\theta_t\|_{L^\infty(\mathbf{R}^2)},$$

so the integral of the second term tends to zero as $R$ tends to infinity. Similar remarks apply to the third and fourth terms. The integral of the last term is

$$\int_{\mathbf{R}^2} (1-\rho) u (\varphi_{tt} - c^2 \varphi_{xx}),$$

but we may restrict the integral to the complement of the ball $B_R$ of radius $R$, since the factor $1-\rho$ ensures that the integrand is zero on $B_R$. Thus the integral is bounded by

$$\|1-\rho\|_{L^\infty(\mathbf{R}^2)} \|u\|_{L^\infty(\mathbf{R}^2)} \|\varphi_{tt} - c^2 \varphi_{xx}\|_{L^1(\mathbf{R}^2 - B_R)}.$$

The first two factors are independent of $R$, while the last tends to zero as $R$ tends to infinity. Thus we see that

$$\int_{\mathbf{R}^2} u (\varphi_{tt} - c^2 \varphi_{xx})$$

tends to zero as $R$ tends to infinity. But the integral is independent of $R$, and must therefore be zero.

The hypothesis that $u$ is bounded, which was used several times in the argument above, can be weakened, but cannot be removed entirely. The difficulty is that there are perfectly good weak solutions, like

$$\exp(x - ct)$$

which are not tempered distributions.

2. Consider the function

$$u(x, y) = \frac{1}{4\pi} \log(x^2 + y^2)$$

as a distribution in the usual way, *i.e.*

$$\langle u, \varphi \rangle = \int_{\mathbf{R}^2} u(x, y) \varphi(x, y) \, dx \, dy.$$

Prove that

$$u_{xx} + u_{yy} = \delta$$

where the derivatives are to be interpreted in the sense of distributions and $\delta$ is the Dirac distribution

$$\langle \delta, \varphi \rangle = \varphi(0, 0).$$

*Solution:* What we need to prove is that

$$\int_{\mathbf{R}^2} u(\varphi_{xx} + \varphi_{yy}) = \varphi(0,0)$$

for all $\varphi \in \mathcal{S}(\mathbf{R}^2)$. This can be proved by splitting the integral into an integral over the ball $B_R$ of radius $R < 1$ and its complement. The integral over $B_R$ is bounded by

$$\|\varphi_{xx} + \varphi_{yy}\| \int_{B_R} |u| = \|\varphi_{xx} + \varphi_{yy}\| \pi R^2 \left(\frac{1}{2} - \log R\right)$$

which tends to zero as $R$ tends to zero. In the integral over the complement of $B_R$ we apply Green's second identity. Since $u_{xx} + u_{yy} = 0$ only the boundary terms survive.

$$\int_{\mathbf{R}^2 - B_R} u(\varphi_{xx} + \varphi_{yy}) = \int_{C_R} \frac{\varphi}{2\pi R}\, ds - \int_{C_R} \frac{\log R}{2\pi} \varphi\, ds.$$

As $R$ tends to zero the first term on the right tends to $\varphi(0,0)$ while the second tends to zero. Adding the integrals over $B_R$ and its complement, we see that

$$\int_{\mathbf{R}^2} u(\varphi_{xx} + \varphi_{yy})$$

tends to $\varphi(0,0)$ as $R$ tends to zero. Since the integral is independent of $R$ it must be identically equal to $\varphi(0,0)$.

3. Prove, by evaluating the integral, that the Poisson Formula

$$u(r,\theta) = \frac{1}{2\pi} \int \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - \varphi) + r^2} f(\varphi)\, d\varphi$$

gives the correct answer when $f$ is constant.
*Solution:* First we make the substitution $\psi = \varphi - \theta$ to get

$$u(r,\theta) = \frac{1}{2\pi} \int \frac{a^2 - r^2}{a^2 - 2ar\cos(\psi) + r^2} f\, d\psi$$

and then the rationalising substitution

$$\cos\psi = \frac{1 - t^2}{1 + t^2} \qquad \sin\psi = \frac{2t}{1 + t^2} \qquad d\psi = \frac{dt}{1 + t^2}$$

which gives

$$u(r,\theta) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{a^2 - r^2}{(a - r)^2 + (a + r)^2 t^2} f\, dt.$$

Finally we make the substitution

$$s = \frac{a+r}{a-r} t$$

to get

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{f\, ds}{1+s^2}$$

This is just equal to $f$.

4. Suppose $(x_P, y_P)$ and $(x'_P, y'_P)$ are points in the unit disc $x^2 + y^2 < 1$ and that the four points $(-1, 0)$, $(x_P, y_P)$, $(x'_P, y'_P)$ and $(1, 0)$ lie on a common circle. Prove that there is a symmetry of the Laplace equation which leaves the disc, the circle and the points $(-1, 0)$ and $(1, 0)$ invariant, while taking $(x_P, y_P)$ to $(x'_P, y'_P)$.
   *Solution:* There is, as shown in class, a Lorentz transformation which takes any three distinct points to any other three distinct points, and therefore there is one which takes $(-1, 0)$, $(1, 0)$ and $(x_P, y_P)$ to $(-1, 0)$, $(1, 0)$ and $(x'_P, y'_P)$. Lorentz transformations take circles or lines to circles or lines and any three distinct points uniquely determine a circle or line. If follows that the Lorentz transformation must leave invariant the circle, which I will call $C$, through $(-1, 0)$, $(x_P, y_P)$, $(x'_P, y'_P)$ and $(1, 0)$. All we have left to show is that the unit circle is also invariant. The unit circle is uniquely determined by the fact that it passes through $(-1, 0)$ and $(1, 0)$ by the angle at which it intersects $C$ at those points. Since the Lorentz transformation leaves points $(-1, 0)$ and $(1, 0)$ and the circle $C$ invariant it must also leave the unit circle invariant.

5. Let $u$ be the solution to the Dirichlet problem

$$u_{xx} + u_{yy} = 0$$

$$u(\cos\theta, \sin\theta) = \begin{cases} -1 & \text{if } -\pi < \theta < 0 \\ +1 & \text{if } 0 < \theta < \pi \end{cases}$$

   Prove that $u$ is constant on each circle passing through the points $(-1, 0)$ and $(1, 0)$. You may use the result of the preceding problem, even if you didn't succeed in proving it.
   *Solution:* Let $\varphi$ be the symmetry from the previous problem and let $\tilde{u} = u \circ \varphi$. $u$ and $\tilde{u}$ satisfy the Laplace equation, because $u$ does and $\varphi$ is a symmetry of the Laplace equation, and satisfy the same boundary conditions, because $\varphi$ preserves the unit circle and the two points $(-1, 0)$ and $(1, 0)$ which separate the positive and negative angles $\theta$.

By the uniqueness of solutions to the Dirichlet problem $u$ and $\tilde{u}$ are therefore the same function. But then

$$u(x_P, y_P) = \tilde{u}(x_P, y_P) = u(x'_P, y'_P).$$

Since $(x_P, y_P)$ were arbitrary points on the circle it follows that $u$ is constant on the circle.