MA 419 Assignment 3

Due 17.00 on 8 December 2006 in the Maths Office

Solutions

1. Solve the initial value problem

$$u(0,x) = x^2 \qquad u_t(0,x) = 0$$

for the wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

on the half line $x \geq 0$ with Dirichlet boundary conditions

$$u(t,0) = 0$$

by the method of reflection.
Solution: Since the boundary conditions are Dirichlet, we need the odd extension

$$\varphi_{\text{odd}}(x) = \begin{cases} -x^2 & \text{if } x < 0 \\ x^2 & \text{if } x > 0 \end{cases}$$
$$\psi(x) = 0.$$

The fundamental solution then gives

$$u = \frac{1}{2}(x+ct) + \frac{1}{2}(x-ct)^2 = x^2 + c^2t^2$$

for $x > ct$ and

$$u = \frac{1}{2}(x+ct) - \frac{1}{2}(x-ct)^2 = 2cxt$$

for $x < ct$.

2. Solve the initial value problem

$$u|_{t=0} = \varphi$$

for the Diffusion equation

$$u_t - ku_{xx} = 0$$

on the interval $[0,l]$ with Neumann boundary conditions at the two ends

$$u_x|_{x=0} = 0 = u_x|_{x=l}$$

by the method of reflection.
Solution: To enforce the Neumann condition at both ends the appropriate extension is

$$\varphi_{NN}(x) = \begin{cases} \varphi(2nl - x) & \text{if } (2n-1)l < x < 2nl \\ \varphi(x - 2nl) & \text{if } 2nl < x < (2n+1)l \end{cases}$$

The solution to the initial value problem

$$u_{NN}|_{t=0} = \varphi_{NN}$$

$$(u_{NN})_t - k(u_{NN})_{xx} = 0$$

on the whole real line is then

$$u_{NN}(t,x) = \int_{-\infty}^{\infty} S(t, x-y)\varphi_{NN}(y)\, dy.$$

Splitting the integral,

$$u_{NN}(t,x) = \sum_{n=-\infty}^{\infty} \int_{(2n-1)l}^{(2n+1)l} S(t, x-y)\varphi_{NN}(y)\, dy.$$

Shifting the variable of integration by $2nl$, and using the fact that φ_{NN} is periodic with period $2l$,

$$u_{NN}(t,x) = \sum_{n=-\infty}^{\infty} \int_{-l}^{l} S(t, x-2nl-y)\varphi_{NN}(y)\, dy.$$

Splitting the integral into an integral over $[-l, 0]$ and an integral over $[0, l]$, and changing the sign of the variable of integration in the former, one obtains

$$u_{NN}(t,x) = \sum_{n=-\infty}^{\infty} \int_{0}^{l} (S(t, x-2nl-y) + S(t, x-2nl+y))\varphi_{NN}(y)\, dy.$$

Let u be the restriction of u_{NN} to the interval $[0, l]$. Then

$$u(t,x) = \int_0^l S_{NN}(t,x,y)\varphi(y)\, dy$$

where

$$S_{NN}(t,x) = \sum_{n=-\infty}^{\infty} (S(t, x-2nl-y) + \sum_{n=-\infty}^{\infty} S(t, x-2nl+y)).$$

3. Solve the same problem by the method of separation of variables. You may omit the hard step, *i.e.* showing that any reasonable initial data φ can be represented as an infinite sum of the appropriate type, but you should calculate the coefficients and state clearly what result you would need to complete the proof.

Solution: As in the case of Dirichlet boundary conditions, we see that the separated solutions are of the form $T(t)X(x)$ where T and X satisfy

$$X''(x) + \lambda X(x) = 0$$

$$T'(t) + k\lambda T(t) = 0$$

for some λ. In order to satisfy the boundary conditions we need

$$\lambda = \frac{n^2\pi^2}{l^2}$$

for an integer l. *Note that, in contrast to the Dirichlet case, there is a nontrivial solution with* $n = 0$. The separated solutions are multiples of

$$\cos\left(\frac{n\pi x}{l}\right)\exp\left(-\frac{kn^2\pi^2 t}{l^2}\right)$$

and we therefore look for a solution of the form

$$u(t,x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right)\exp\left(-\frac{kn^2\pi^2 t}{l^2}\right).$$

To satisfy the boundary conditions we need

$$\sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{l}\right) = \varphi(x).$$

Multiplying by $\cos\left(\frac{m\pi x}{l}\right)$ and integrating leads to

$$A_n = \begin{cases} \frac{1}{l}\int_0^l \varphi(x)\,dy & \text{if } n = 0 \\ \frac{2}{l}\int_0^l \varphi(y)\cos\left(\frac{n\pi y}{l}\right)\,dy & \text{if } n > 0 \end{cases}.$$

If we assume that indeed

$$\frac{1}{l}\int_0^l \varphi(x)\,dy + \sum_{n=1}^{\infty}\frac{2}{l}\int_0^l \left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi y}{l}\right)\varphi(y)\,dy = \varphi(x)$$

then

$$u(t,x) = \frac{1}{l}\int_0^l \left[1 + 2\sum_{n=1}^{\infty} \exp\left(-\frac{kn^2\pi^2 t}{l^2}\right)\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi y}{l}\right)\right]\varphi(y)\,dy$$

is a solution to the Diffusion Equation satisfying both the initial conditions and the boundary conditions.

4. Prove conservation of energy in the form

$$\frac{d}{dt}\int_0^{\infty}\left(u_t^2 + c^2 u_x^2\right)dx = 0$$

for solutions of the wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

with either Dirichlet or Neumann boundary conditions. Does the result hold for the Robin condition?
Solution: Differentiating under the integral sign, and using the fact that

$$\partial_t\left(u_t^2 + c^2 u_x^2\right) = \partial_x\left(2c^2 u_t u_x\right),$$

for any solution u of the wave equation, we find

$$\begin{aligned}\frac{d}{dt}\int_0^X \left(u_t^2 + c^2 u_x^2\right)dx &= \int_0^X \partial_t\left(u_t^2 + c^2 u_x^2\right)dx\\ &= \int_0^X \partial_x\left(2c^2 u_t u_x\right)dx\\ &= 2c^2 u_t(t,X)u_x(t,X) - 2c^2 u_t(t,0)u_x(t,0).\end{aligned}$$

Note that

$$2cu_t(X)u_x(X) \le u_t^2(t,X) + c^2(t,X).$$

The right hand side is integrable by hypothesis. It follows that the right hand side is as well and thus that there is a sequence X_n tending to infinity such that

$$\lim_{N\to\infty} 2cu_t(X)u_x(X) = 0.$$

Thus

$$\frac{d}{dt}\int_0^{\infty}\left(u_t^2 + c^2 u_x^2\right)dx = \frac{d}{dt}\lim_{N\to\infty}\int_0^{X_N}\left(u_t^2 + c^2 u_x^2\right)dx = -2c^2 u_t(t,0)u_x(t,0).$$

If u satisfies either the Dirichlet or Neumann conditions then the right hand side is zero.

If u satisfies the Robin Condition

$$(u_x + au)|_{x=0} = 0$$

then the derivative is not zero in general. There is a partial substitute

$$\frac{d}{dt}\left(ac^2u(t,0)^2 + \int_0^\infty \left(u_t^2 + c^2u_x^2\right)dx\right) = 0.$$

This is only useful if $a > 0$, since it is only in this case that we get an upper bound for

$$\int_0^\infty \left(u_t^2 + c^2u_x^2\right)dx$$

in terms of the initial data.