MA 419 Assignment 2

Due 15 November 2006

Solutions

1. Compute $f \star g$ where f and g are defined by

$$f(x) = \exp(-|x|)$$

and

$$g(x) = \begin{cases} 0 & \text{if } x < -1 \\ 1 & \text{if } -1 \le x \le 1 \\ 0 & \text{if } 1 < x \end{cases} .$$

Solution: By definition,

$$(f \star g)(x) = \int_{-\infty}^{+\infty} f(x-y)g(y)\,dy = \int_{-1}^{+1} \exp(-|x-y|)\,dy.$$

There are three cases to consider. If $x < -1$ then

$$(f \star g)(x) = \int_{-1}^{+1} \exp(x-y)\,dy = \exp(x+1) - \exp(x-1).$$

If $-1 \le x \le 1$ then

$$\begin{aligned}(f \star g)(x) &= \int_{x}^{+1} \exp(x-y)\,dy + \int_{-1}^{x} \exp(y-x)\,dy \\ &= 2 - \exp(x-1) - \exp(-x-1).\end{aligned}$$

If $1 < x$ then

$$(f \star g)(x) = \int_{-1}^{+1} \exp(y-x)\,dy = \exp(-x+1) - \exp(-x-1).$$

2. The solution of the initial value problem for the homogeneous Wave Equation

$$u_{tt} - c^2 u_{xx} = 0 \qquad u|_{t=0} = \varphi \qquad u_t|_{t=0} = \psi$$

in the special case $\varphi = 0$ can be written in the form

$$u(t, \cdot) = K(t, \cdot) \star \psi$$

for an appropriate function K. What is this K?
Solution:

$$K(t,x) = \begin{cases} 0 & \text{if } x < -ct \\ \frac{1}{2c} & \text{if } -ct \le x \le ct \\ 0 & \text{if } ct < x \end{cases}$$

3. A function f is said to be α-Hölder continuous if, for every x, there is an L and a $\delta > 0$ such that

$$|f(y) - f(x)| < L|y - x|^\alpha$$

whenever $|y - x| < \delta$. Show that if there is a positive α such that f is α-Hölder continuous then f is continuous. Show that if $f' \in L^p(\mathbf{R})$ for some $1 \le p \le \infty$ then f is α-Hölder continuous for

$$\alpha = 1 - \frac{1}{p}.$$

Solution: Assume f is α-Hölder continuous. Then, by definition, there is an L and a $\theta > 0$ such that

$$|f(y) - f(x)| < L|y - x|^\alpha$$

whenever $|y - x| < \theta$. For $\epsilon > 0$, set

$$\delta = \min(\theta, (\epsilon/L)^{1/\alpha}).$$

If $|y - x| < \delta$ then

$$|f(y) - f(x)| \le L|y - x|^\alpha < L\delta^\alpha \le \epsilon.$$

Thus α-Hölder continuity implies continuity.

Now assume $f' \in L^p(\mathbf{R})$. Assume, without loss of generality, that $x \le y$. By the Fundamental Theorem of Calculus,

$$f(y) - f(x) = \int_x^y f'(s)\, ds.$$

This can also be written as

$$f(y) - f(x) = \int_{-\infty}^{+\infty} f'(s) g(s)\, ds$$

where

$$g(s) = \begin{cases} 0 & \text{if } s < x \\ 1 & \text{if } x \le s \le y \\ 0 & \text{if } y < s \end{cases}$$

By Hölder's inequality,

$$|f(y) - f(x)| \le \|f'\|_{L^p(\mathbf{R})} \|g\|_{L^q(\mathbf{R})} = L|x - y|^\alpha$$

where

$$L = \|f'\|_{L^p(\mathbf{R})}$$

and $q = 1/\alpha$. Note that $\frac{1}{p} + \frac{1}{q} = 1$, as required for Hölder's inequality. Thus $f' \in L^p(\mathbf{R})$ implies α-Hölder continuity.

4. Solve the Diffusion Equation

$$u_t - ku_{xx}$$

with initial data

$$u(0,x) = e^{\lambda x}.$$

Solution: We use the fundamental solution

$$u(t,x) = (4\pi kt)^{-1/2} \int_{-\infty}^{+\infty} \exp\left(-\frac{(x-y)^2}{4kt}\right) u(0,y)\,dy$$

with the given initial data. Making the substitution

$$s = (4\pi kt)^{-1/2}\,(x - y + 2k\lambda t)$$

gives

$$u(t,x) = \exp(\lambda(x + k\lambda t)) \int_{-\infty}^{+\infty} \exp(-\pi s^2)\,ds = \exp(\lambda(x + k\lambda t)).$$

5. With S defined, as in lecture, by

$$S(t,x) = (4\pi kt)^{-1/2} \exp\left(-\frac{x^2}{4kt}\right),$$

prove that for all positive s and t

$$S(s+t,\cdot) = S(s,\cdot) \star S(t,\cdot).$$

Solution: By the definition of convolution,

$$(S(s,\cdot) \star (S(t,\cdot))(x) = \int_{-\infty}^{+\infty} S(s,x-y)S(t,y)\,dy$$

For brevity, call the right hand side I. Substituting in the definition of S,

$$I = (4\pi ks)^{-1/2}(4\pi kt)^{-1/2} \int_{-\infty}^{+\infty} \exp\left(-\frac{(x-y)^2}{4ks} - \frac{y^2}{4kt}\right) dy$$

Making the substitution

$$z = (4\pi kst(s+t))^{-1/2}((s+t)y - tx),$$

we find

$$I = (4\pi k(s+t))^{-1/2} \exp\left(-\frac{x^2}{4k(s+t)}\right) \int_{-\infty}^{+\infty} \exp(-\pi z^2)\,dz = S(s+t,x).$$