MA 419 Assignment 1

Due Wednesday 25 October 2006

Solutions

1. For each of the following, say whether it is a scalar equation or system, give the order, and state whether it is linear or non-linear. If linear, state whether it is homogeneous or inhomogenous.

   (a) The 1+3 Dimensional Wave Equation

   $$u_{tt} - u_{xx} - u_{yy} - u_{zz} = 0$$

   (b) The 1+1 Dimensional Klein-Gordon Equation

   $$u_{tt} - u_{xx} = u$$

   (c) The 1+3 Dimensional Eikonal Equation

   $$u_t^2 - u_x^2 - u_y^2 - u_z^2$$

   (d) The Incompressible Euler Equations

   $$\begin{aligned} u_t + uu_x + vu_y + wu_z + p_x &= 0 \\ v_t + uv_x + vv_y + wv_z + p_y &= 0 \\ w_t + uw_x + vw_y + ww_z + p_z &= 0 \\ u_x + v_y + w_z &= 0 \end{aligned}$$

   (e) The Euler-Tricomi Equation

   $$u_{xx} = xu_{yy}$$

   (f) The Korteweg-de Vries Equation

   $$u_t + u_{xxx} - 6uu_x = 0$$

   Solution:

   (a) The 1+3 Dimensional Wave Equation is a second order linear homogeneous scalar equation.

   (b) The 1+1 Dimensional Klein-Gordon Equation is also a second order linear homogeneous scalar equation.

(c) The 1+3 Dimensional Eikonal Equation is a first order nonlinear scalar equation.

(d) The Incompressible Euler Equations are a first order nonlinear system.

(e) The Euler-Tricomi Equation is a second order linear homogeneous scalar equation.

(f) The Korteweg-de Vries Equation is a third order nonlinear system.

2. Find all solutions of the first order linear homogeneous equation

$$u_t - xu_x = 0$$

Solution: The characteristic equations

$$\frac{dt}{ds} = 1 \quad \frac{dx}{ds} = -x$$

have solutions

$$t = s + c_1 \quad x = c_2 \exp(-s)$$

so the characteristic curves are

$$x = a \exp(-t).$$

The parameter $a = c_2 \exp(c_1)$ identifies which curve we are looking at. We lose nothing by taking $c_1 = 0$ and parametrizing the curves by $s = t$. In terms of the a, s variables

$$u_t - xu_x = u_s$$

so u is a function only of a,

$$u(t, x) = f(a) = f(x \exp(t)).$$

When $t = 0$, $x \exp(t) = x$, so

$$u(0, x) = f(x)$$

and hence

$$u(t, x) = u(0, x \exp(t)).$$

3. Find all solutions of the first order linear homogeneous equation

$$u_t - xu_x + tu = 0$$

Solution: The same change of variables works. In the a, s variables the equation becomes

$$u_s + su = 0$$

with solutions of the form

$$u = f(a) \exp(-s^2/2) = f(x \exp(t)) \exp(-t^2/2).$$

In terms of the initial conditions,

$$u(t, x) = u(0, x \exp(t)) \exp(-t^2/2).$$

4. Solve the initial value problem for the Wave Equation

$$u_{tt} - u_{xx} = 0$$

with initial data

$$u(0, x) = 0 \qquad u_t(0, x) = \frac{1}{1 + x^2}.$$

Solution: Using the explicit solution

$$u(t, x) = \frac{1}{2}u(x + ct) + \frac{1}{2}u(x + ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} u_t(0, y)\, dy$$

and evaluating the resulting integral,

$$u(t, x) = \frac{\arctan(x + ct) - \arctan(x - ct)}{2c}.$$