MA 419 Study Guide 2007

April 18, 2007

1 General Comments

The exam is not meant to be evil. Anyone who has consistently put in the effort to attend the lectures, attempted to solve the problems on the assignments,[1] *read the posted solutions*, and revised his or her notes should have no difficulty in obtaining a passing mark. Obtaining a first is probably quite hard, but certainly not impossible.

2 What you need to know

The following is not an exact description of the content of the exam by any means. Rather it is a list of the main things you should know how to do. Some of these are not on the exam. Some questions which are on the exam cover topics not on the list. On the whole, though, you should be in good shape for the exam if you concentrate your revision primarily on what is listed here.

1. Explicit solutions.
   You must know the various formulae for explicit solutions of the three main examples

[1]If you made a serious attempt, and failed to get one or more problems then you are probably in much better shape than someone who merely transcribed correct answers from someone else's assignment.

(a) Wave Equation (you need only remember the explicit solution in one dimension)

(b) Diffusion Equation (in one or more dimensions)

(c) Laplace Equation in a disc or half plane, *i.e.* the two versions of the Poisson formula.

You must be able to apply these for given initial or boundary data. You need not derive them.

2. Boundary and Initial Conditions
   Where an explicit solution is the solution to an initial value or boundary value problem you must know in what sense it solves the initial or boundary conditions, *e.g.* L^p convergence or pointwise convergence almost everywhere or by extending continuously to the boundary.

3. Existence and Uniqueness.
   You must know the statements of the various existence and uniqueness theorems for the three main example equations. You should know the main ideas of the proofs, except for the proof of uniqueness for the Diffusion Equation. You should also know how to apply the uniqueness theorem to derive properties of the solution, *e.g.* to derive symmetries of the solution from symmetries of the data. You may be required to prove similar theorems for unfamiliar equations, using the same methods.

4. Regularity and Stability
   You must know how the estimates on solutions to the Diffusion Equation were obtained, *i.e.* differentiation of the explicit solution and application of Young's Identity. You need not memorise the resulting estimates, but you should be prepared to derive similar estimates for other equations.

5. The Divergence Theorem.
   You must know the Divergence Theorem. More importantly, you must know its consequences for Partial Differential Equations:

   (a) Green's First and Second Identities

   (b) Conservation Laws

You should know how to use Green's First and Second Identities to prove properties of the Laplace Equation such as the Maximum Principle (Strong or Weak) and uniqueness of solutions to boundary value problems. You should know to use the Divergence Theorem to obtain conservation laws.

6. Method of Reflection.
   You should know which reflections satisfy which boundary conditions and why. You should also know how multiple reflections combine to produce periodicity.

7. Notions of solution.
   You must know the definitions of the following.

   (a) Strong solutions, a.k.a. Classical solutions

   (b) Weak solutions

   (c) Distribution solutions

   (d) Shock solutions (only in the case of Burgers' Equation)

   You should be able to identify in which sense a given function is a solution to a particular equation. You should also know which statements imply which others, *e.g.* strong solutions are necessarily weak solutions. You could be asked for examples of functions which are solutions in one sense but not another.

8. The Method of Characteristics for first order scalar equations.

   (a) Linear equations (homogeneous or inhomogeneous)

   (b) Burgers' equation

9. Symmetries of the Laplace Equation

   (a) The relation between symmetries and Lorentz transformations.

   (b) The relations between spacelike vectors and circle or lines and between lightlike vectors and metrics

   (c) The relation between angles of intersection and inner products

   (d) Constructing transformations which send a given configuration of points, lines and circles to another given configuration

You should be able to work out specific examples. You needn't remember the proofs, except in so far as they are helpful in working out examples. I have half-finished notes on this material, which I will post in a week or two.

10. Basic Harmonic Analysis

    (a) Inequalities: Hölder, Young, Minkowski

    (b) Lebesgue points

    (c) Convolution, both for functions and distributions

    (d) The δ distribution

    (e) Differentiation of Distributions

    You must know the basic definitions. You need not know any proofs. You should know how to use these definitions in examples.

3 The Structure of the Exam

There are nine questions, each worth 20 points. You might be asked to attempt five or you might be asked to attempt six. At the moment the instructions say to attempt six, but I have a request in with the External Examiner to change it to five. Whatever his reply, it will certainly be true that you can skip problems, and you will probably want to. It is worth spending some time in choosing which problems to attempt. Some look positively evil at first glance, but are not very hard after a closer examination. Some look straightforward, and are, but are time consuming. You can afford to do a couple of such problems, but if you do too many you will not have time for the remaining problems. All questions have multiple parts, and the number of points for each part is marked on the exam.

It is difficult to classify some of the questions, but what follows is a very rough breakdown. Note that I have indicated similar problems from the assignments, where applicable. These are often harder than the exam problems. Of course, on the exam you have the disadvantages of limited time and no access to your notes.

1. Straightforward Computation.
   By straightforward I do not mean easy. Some are easy, some are hard.

What I mean by straightforward is that you need to know a formula which you were told that you should memorise and that you then need to perform a calculation of a reasonably routine nature, *e.g.* partial differentiation, integration by partial fractions or a standard substitution, solving a linear or separable ordinary differential equation. In other words, these are problems for which there is no "trick" to be found, but the calculations may require some care and absorb some amount of time. By my calculation these make up 70 points worth of questions. Examples from the assignments of the sort of question I have in mind are 1.2, 1.3, 1.4, 2.1, 2.4, 2.5, 4.3, 5.2, 6.2 and 6.3.

2. Proofs
   These are normally not proofs of statements from lecture or from the assignments, although some are. With one or two exceptions, though, they follow very closely proofs of statements that did appear in lecture or on an assignment. For example, you might be asked to prove the analogue for the three dimensional Laplace Equation of a statement which I proved, or you proved on an assignment, for the two dimensional equation. You could also be asked to prove something for an equation which you haven't seen before, but which is very similar to one you have seen. Most of these problems involve some element of straightforward calculation as well, but, unlike the previous type, the emphasis is on the theory rather than the calculation itself. By my calculation these make up 57 points worth of questions. Examples from the assignments of the sort of question I have in mind are 2.3, 3.4, 4.1, 4.2, 4.4, 4.5, 5.1 5.4, 5.5, 6.1, 6.4, and 6.5.

3. Derivation of (Unseen) Formulae
   This is roughly the same category as "Proofs" above, except that you are not told explicitly the statement you are meant to prove. The classic example would be find an integral representation to the solution of equation X subject to the boundary conditions Y, but other things are possible. By my calculation these make up 10 points worth of questions. Examples from the assignments of the sort of question I have in mind are 3.1, 3.2, 3.3 and 5.3.

4. Interpretation of results
   These are questions about understanding the results of a calculation, *e.g.* in what sense your solution from an earlier part solves the equation

or boundary conditions which were specified, or what an example tells you about existence or uniqueness. These questions have short answers and more or less binary marking schemes. Either you get them or you don't. You could also be asked to interpret a formula you obtained, *e.g.* as a convolution. By my calculation these make up 18 points worth of questions. There are no problems on the assigments which are purely of this type. Sometimes the final sentence of an assigned problem asks such a question.

5. Statements
   These simply ask you to repeat some result from lecture without proof. I don't much like such questions and tend to avoid asking them. Where I do ask them they often serve as hints about other parts of the same problem. For example, if I asked you for the statement of the averaging property of harmonic functions it would be very likely that some other part of the same problem either requires this property or is much more easily done using this property than by some alternative method. By my calculation these make up 16 points worth of questions. Examples of the sort of question I have in mind are entirely absent from the assignments, because they would serve no purpose on an assignment where you have free access to your notes.

6. Examples
   By this I mean questions which directly ask for an example, like "give an example of an X which is not a Y" or, more subtly, questions whose answer requires a counter-example, like "do all solutions of X have property Y?" In all cases, the required example was worked out either in lecture or on an assignment. These questions just require the example itself, and a brief indication of why it has the required property, not a long calculation.[2] By my calculation these make up 9 points worth of questions. Examples of the sort of question I have in mind are entirely absent from the assignments, again because they would serve no purpose.

[2] Where a long calculation and not the example is the point of the problem I have listed the problem in the "Straightforward Calculations" section.