MA 419 Lecture Notes
Hilary Term 2007
Lorentz Transformations and Symmetries of the Laplace Equation

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1 Stereographic Projection

To every point on the unit sphere

$$x_S^2 + y_S^2 + z_S^2 = 1$$

except for the North Pole $(0, 0, 1)$ we associate a point in the plane

$$P = (x_P, y_P)$$

as follows. The line through $(0, 0, 1)$ and

$$S = (x_S, y_S, z_S)$$

intersects the xy-plane in the single point

$$\left(\frac{x_S}{1 - z_S}, \frac{y_S}{1 - z_S}, 0\right).$$

We take the first two coordinates to be x_P and y_P,

$$x_P = \frac{x_S}{1 - z_S},$$
$$y_P = \frac{y_S}{1 - z_S}.$$

Similarly, the line through $(0, 0, 1)$ and $(x_P, y_P, 0)$ intersects the unit sphere in two points, one of which is $(0, 0, 1)$ and the other of which is (x_S, y_S, z_S),

$$x_S = \frac{2x_P}{x_P^2 + y_P^2 + 1},$$
$$y_S = \frac{2y_P}{x_P^2 + y_P^2 + 1},$$
$$z_S = \frac{x_P^2 + y_P^2 - 1}{x_P^2 + y_P^2 + 1}.$$

In order to make the correspondence complete we add a "point at infinity" to the plane to obtain a set which we call the extended plane.

2 Projectivisation

Next we introduce projective coordinates t_L, x_L, y_L, z_L in in $\mathbf{R}^3$,

$$x_L = x_S/t_S,$$
$$y_L = y_S/t_S,$$
$$z_L = z_S/t_S.$$

The choice of t_L, x_L, y_L, z_L corresponding to a point $(x_S, y_S, z_S) \in \mathbf{R}^3$ is, of course, far from unique. Any nonzero point on the same line through the origin in $\mathbf{R}^4$ will give the same point in $\mathbf{R}^3$. Conversely, two points in $\mathbf{R}^4$ give the same point in $\mathbf{R}^3$ if and only if they lie on a line through the origin.

The correspondence between points in $\mathbf{R}^3$ and lines through the origin in $\mathbf{R}^4$ is not, however complete. Lines in the hyperplane $t_L = 0$ do not correspond to points. We may obtain a complete correspondence by adjoining a "plane at infinity." The resulting set is called real projective three space and denoted $\mathbf{RP}^3$.

The preceding construction is quite general, but here we need it only for points on the unit sphere

$$x_S^2 + y_S^2 + z_S^2 = 1.$$

These correspond to lines through the origin in the cone

$$t_L^2 - x_L^2 - y_L^2 - z_L^2 = 0.$$

In this case there is no need to introduce the plane at infinity, since there are no such lines in the hyperplane $t_L = 0$.

Combining the ideas of Stereographic Projection and Projectivisation, we identify points in the extended plane with lines through the origin in $\mathbf{R}^4$ which lie in the cone. In coordinates,

$$\begin{aligned} x_P &= \frac{x_L}{t_L - z_L}, \\ y_P &= \frac{y_L}{t_L - z_L} \end{aligned}$$

and

$$\begin{aligned} t_L &= s\frac{x_P^2 + y_P^2 + 1}{2} \\ x_L &= s x_P, \\ y_L &= s y_P, \\ z_L &= s\frac{x_P^2 + y_P^2 + 1}{2} \end{aligned}$$

where s is a nonzero constant which may be chosen arbitrarily. The point at infinity corresponds to the lines through the point

$$N = (1, 0, 0, 1).$$

3 Minkowski Inner Product

It is natural at this point to introduce the Minkowski inner product on $\mathbf{R}^4$,

$$\langle L_1, L_2 \rangle = t_{L1}t_{L2} - x_{L1}x_{L2} - y_{L1}y_{L2} - z_{L1}z_{L2}$$

where

$$L_1 = (t_{L1}, x_{L1}, y_{L1}, z_{L2})$$

and

$$L_2 = (t_{L2}, x_{L2}, y_{L2}, z_{L2}).$$

In terms of this inner product, L lies in the cone if and only if its inner product with itself is zero. The Minkowski inner product is nondegenerate, in the sense that for any $L_1 \neq 0$ there is an L_2 such that $\langle L_1, L_2 \rangle \neq 0$.

Following the standard terminology of Special Relativity, a nonzero vector in $\mathbf{R}^4$ will be called timelike if its inner product with itself is positive, lightlike if the inner product is zero and spacelike if the inner product is negative. Lines through the origin are called timelike, lightlike or spacelike according to the character of their nonzero vectors. The combination of $\mathbf{R}^4$ with the Minkowski inner product is called Minkowski space. sectionMetrics For nonzero vectors X, L_1 and L_2 with L_1 and L_2 lightlike we define $d_X^2(L_1, L_1)$ by

$$d_X^2(L_1, L_2) = \frac{2\langle L_1, L_2 \rangle}{\langle L_1, X \rangle \langle X, L_2 \rangle}$$

provided neither L_1 nor L_2 is orthogonal to X. If $d_X^2(L_1, L_1)$ is nonnegative then we define $d_X(L_1, L_1)$ to be its square root.

Note that

$$d_X^2(s_1 L_1, s_2 L_2) = d_X^2(L_1, L_2)$$

for any nonzero s_1 and s_2, so d_X^2 and d_X can be considered as functions on the extended plane, or on a subset thereof. Though it's far from obvious, d_X is a metric on the space on which it is defined. This will be proved later. Multiplying X by a nonzero factor does change d_X^2 and d_X, but in a very simple way,

$$d_{sX}^2(L_1, L_1) = s^{-2} d_X^2(s_1 L_1, s_2 L_2),$$

$$d_{sX}(L_1, L_1) = |s|^{-1} d_X(s_1 L_1, s_2 L_2).$$

The ratio between different metrics is given by

$$\frac{d_X(L_1, L_2)}{d_Y(L_1, L_2)} = \sqrt{\frac{\langle L_1, Y\rangle \langle Y, L_2\rangle}{\langle L_1, X\rangle \langle X, L_2\rangle}}.$$

For nearby points we have

$$\lim_{L_2 \to L_1} \frac{d_X(L_1, L_2)}{d_Y(L_1, L_2)} = \left|\frac{\langle L_1, Y\rangle}{\langle L_1, X\rangle}\right|.$$

4 Planar and Spherical Metrics

If we take

$$X = N = (1, 0, 0, 1)$$

then it is convenient to express the arguments of d_X^2 in terms of planar coordinates.

$$L_j = \left(\frac{x_{Pj}^2 + y_{Pj}^2 + 1}{2}, x_{Pj}, y_{Pj}, \frac{x_{Pj}^2 + y_{Pj}^2 - 1}{2}\right).$$

Here I have used the fact that the value of $d_X^2(L_1, L_2)$ depends only on the lines through L_1 and L_2 in order to choose convenient representatives, specifically those with $t_{Lj} - z_{Lj} = 1$. Then we can calculate the inner products

$$\langle N, L_j\rangle = 1$$

$$2\langle L_1, L_2\rangle = (x_{P1} - x_{P2})^2 + (y_{P1} - y_{P2})^2$$

so

$$d_N^2(L_1, L_1) = (x_{P1} - x_{P2})^2 + (y_{P1} - y_{P2})^2.$$

Thus d_N is just the Euclidean distance between the points P_1 and P_2 in the plane.

If

$$X = S = (1, 0, 0, 0)$$

then it is more convenient to express the arguments of d_X^2 in terms of spherical coordinates

$$L_j = t_{Lj}(1, x_{Sj}, y_{Sj}, z_{Sj})$$

rather than planar coordinates. Again I have used the fact that $d_X(L_1, L_2)$ depends only on the lines through L_1 and L_2 to choose convenient representative points, this time by choosing $t_{Lj} = 1$. Then

$$\langle L_j, S\rangle = 1$$

$$2\langle L_1, L_2\rangle = 2 - 2x_{S1}x_{S2} - 2y_{S1}y_{S2} - 2z_{S1}z_{S2}.$$

Since

$$x_{Sj}^2 + y_{Sj}^2 + z_{Sj}^2 = 1$$

we can rewrite this as

$$2\langle L_1, L_2\rangle = (x_{S1} - x_{S2})^2 + (y_{S1} - y_{S2})^2 + (z_{S1} - z_{S2})^2.$$

From this we see that

$$d_S^2(L_1, L_2) = (x_{S1} - x_{S2})^2 + (y_{S1} - y_{S2})^2 + (z_{S1} - z_{S2})^2$$

and hence that $d_S(L_1, L_2)$ is the Euclidean distance between the points S_1 and S_2 on the unit sphere.

5 Circles

The point P_2 is at a distance r from the point P_1 if and only if

$$d_N(L_1, L_2) = r^2.$$

This happens if and only if

$$2\langle L_1, L_2\rangle - r^2 \langle L_1, N\rangle \langle N, L_2\rangle = 0.$$

Equivalently $\langle C, L_2 \rangle = 0$ where

$$C = L_1 - \frac{r^2}{2} \langle L_1, N \rangle N.$$

Since

$$\langle C, C \rangle = -r^2 \langle L_1, N \rangle^2$$

C is necessarily a spacelike vector.

$$\langle C, N \rangle = \langle L_1, N \rangle,$$

so r may be recovered from C by

$$r^2 = -\frac{\langle C, C \rangle}{\langle C, N \rangle^2}.$$

and location of the centre L_1 is

$$L_1 = C - \frac{1}{2} \frac{\langle C, C \rangle}{\langle C, N \rangle} N.$$

6 Circles and Lines

As shown in the preceding section, circles in the plane correspond to spacelike vectors in Minkowski space. More precisely, for every circle in the plane there is a spacelike vector C in Minkowski space such that $P = (x_P, y_P)$ lies on the circle if and only if

$$\langle C, L \rangle = 0,$$

where $L = (t_L, x_L, y_L, z_L)$ are related in the usual way by stereographic projection and projectivisation. It is clear from the equation that any nonzero multiple of C will define the same circle, so we may consider circles in the plane as defining spacelike lines in Minkowski space. Since the radius and centre are given by

$$\sqrt{-\frac{\langle C, C \rangle}{\langle C, N \rangle^2}}$$

and

$$C - \frac{1}{2} \frac{\langle C, C \rangle}{\langle C, N \rangle} N$$

this correspondence is one to one, but it is not onto.

Consider an arbitrary spacelike vector

$$C = (\tau, \xi, \eta, \zeta).$$

If

$$L = s(\frac{x_P^2 + y_P^2 + 1}{2}, x_P, y_P, \frac{x_P^2 + y_P^2 - 1}{2})$$

then

$$\langle C, L \rangle = \frac{s}{2}(\tau - \zeta)(x_P^2 + y_P^2) - 2\xi x_P - 2\eta y_P - \tau - \zeta$$

If $\tau = \zeta$, *i.e.* if $\langle C, N \rangle = 0$, then the equation $\langle C, L \rangle = 0$ determines a line in the extended plane. Since $\langle C, N \rangle = 0$ we should consider the point at infinity to lie on this line. On the other hand, if $\tau \neq \zeta$ then $\langle C, L \rangle = 0$ determines the circle

$$(x_P - x_c)^2 + (y_P - y_c)^2 + \frac{\tau^2 - \xi^2 - \eta^2 - \zeta^2}{(\tau - \zeta)^2} = 0$$

with centre

$$(x_c, y_c) = \left(\frac{\xi}{\tau - \eta}, \frac{\eta}{\tau - \zeta} \right)$$

and radius

$$\sqrt{-\frac{\tau^2 - \xi^2 - \eta^2 - \zeta^2}{(\tau - \zeta)^2}}$$

which is easily seen to agree with the formulae of the previous section.

7 Angles and Intersections

In Euclidean space the angle between two nonzero vectors X and Y is given by

$$\cos\theta = \frac{\langle X,Y\rangle}{\sqrt{\langle X,X\rangle\langle Y,Y\rangle}}.$$

In Minkowski space this definition is problematic. The quantity under the square root sign need not be positive. In fact it will positive if and only if X and Y are either both spacelike or both timelike. Even if this is the case, it can still happen that the quotient is outside the range $[-1,+1]$. There is, however, and important case in which the angle is well-defined and carries useful geometric information.

Suppose C_1 and C_2 are spacelike vectors. These correspond, as we saw in the preceding sections, to circles of radii

$$r_1 = \sqrt{-\frac{\langle C_1,C_1\rangle}{\langle C_1,N\rangle^2}}$$

and

$$r_2 = \sqrt{-\frac{\langle C_2,C_2\rangle}{\langle C_2,N\rangle^2}}$$

The centres of these circles are located at the points in the plane corresponding to the lightlike vectors

$$L_1 = C_1 - \frac{1}{2}\frac{\langle C_1,C_1\rangle}{\langle C_1,N\rangle}N$$

and

$$L_2 = C_2 - \frac{1}{2}\frac{\langle C_2,C_2\rangle}{\langle C_2,N\rangle}N.$$

The distance between the centres is then

$$r_{12} = d_N(L_1,L_2) = \sqrt{\frac{2\langle L_1,L_2\rangle}{\langle L_1,N\rangle\langle N,L_2\rangle}}.$$

The inner products are easily calculated.

$$\langle L_1,N\rangle = \langle C_1,N\rangle,$$

$$\langle L_2,N\rangle = \langle C_2,N\rangle$$

and

$$2\langle L_1,L_2\rangle = 2\langle C_1,C_2\rangle - \frac{\langle C_1,C_1\rangle\langle C_2,N\rangle}{\langle C_1,N\rangle} - \frac{\langle C_2,C_2\rangle\langle C_1,N\rangle}{\langle C_2,N\rangle}$$

so

$$d_N^2(L_1,L_2) = 2\frac{\langle C_1,C_2\rangle}{\langle C_1,N\rangle\langle N,C_2\rangle} + r_1^2 + r_2^2.$$

or, equivalently,

$$r_{12}^2 = r_1^2 + r_2^2 \pm 2\mu r_1 r_2$$

where

$$\mu = -\frac{\langle C_1,C_2\rangle}{\sqrt{\langle C_1,C_1\rangle\langle C_2,C_2\rangle}}$$

and the $\pm$ sign is to be interpreted as positive if $\langle C_1,N\rangle$ and $\langle C_2,N\rangle$ are of the same sign and negative if they are of opposite signs.

If $\mu < -1$ or $\mu > 1$ then either

$$r_{12} < |r_1 - r_2|$$

or

$$r_{12} > r_1 + r_2$$

and the circles do not intersect. If $\mu = \pm 1$ then either

$$r_{12} = |r_1 - r_2|$$

or

$$r_{12} = r_1 + r_2$$

and the two circles have a common tangent line. The circles intersect in a single point, which must be that corresponding to

$$L_3 = \sqrt{-\langle C_2,C_2\rangle}C_1 + \sqrt{-\langle C_1,C_1\rangle}C_2$$

if $\mu = -1$ or

$$L_4 = \sqrt{-\langle C_2,C_2\rangle}C_1 - \sqrt{-\langle C_1,C_1\rangle}C_2$$

if $\mu = 1$, as this point is on both circles.

If $-1 < \mu < 1$ then

$$|r_1 - r_2| < r_{12} < r_1 + r_2$$

and the circles intersect in two points. The angle of intersection can be determined by use of the Law of Cosines applied to a triangle with vertices at the centres of the two circles and at one of the points of intersection. This shows that the angle of intersection of the two circles is θ where

$$\cos\theta = \pm\mu,$$

the determination of the sign $\pm$ being as before.

Similar remarks apply when one or both of the vectors C corresponds to a line. The details are left as an exercise.

8 Lorentz Transformations

Linear transformations of Minkowski space which preserve its inner product are called Lorentz Transformations. Since such transformations necessarily take lightlike lines to lightlike lines we can think of a Lorentz transformation as acting on the extended plane. Since Lorentz transformations take spacelike likes to spacelike lines the corresponding action on the extended plane takes circles or lines to circles or lines. In addition, since Lorentz transformations preserve inner products, the angle of intersection, if any, of a pair of lines or circles is also unchanged. For historical reasons this property is referred to a conformal invariance.

If X, L_1 and L_2 are transformed to X', L_1' and L_2' then

$$\begin{aligned} d_X^2(L_1, L_2) &= \frac{\langle L_1, L_2\rangle}{\langle L_1, X\rangle \langle X, L_2\rangle} \\ &= \frac{\langle L_1', L_2'\rangle}{\langle L_1', X'\rangle \langle X', L_2'\rangle} = d_{X'}^2(L_1', L_2'). \end{aligned}$$

It follows, in particular, that Lorentz transformations which leave N invariant correspond to rigid motions of the plane, *i.e.* they leave the Euclidean distance between pair of points unchanged. Similarly, Lorentz transformations which leave S invariant correspond to rigid motions of the unit sphere, *i.e.* rotations or reflections of $\mathbf{R}^3$ or the composition of a rotation and a reflection.

9 Rotations

The one parameter family of Lorentz transformations

$$\begin{aligned} t_L' &= t_L \\ x_L' &= x_L\cos\theta - y_L\sin\theta \\ y_L' &= x_L\sin\theta + y_L\cos\theta \\ z_L' &= z_L \end{aligned}$$

preserves both N and S. A quick calculation gives

$$\begin{aligned} x_P' &= \frac{x_L'}{t_L' - z_L'} \\ &= \frac{x_L\cos\theta - y_L\sin\theta}{t_L - z_L} \\ &= \frac{x_L}{t_L - z_L}\cos\theta - \frac{y_L}{t_L - z_L}\sin\theta \\ &= x_P\cos\theta - y_P\sin\theta, \end{aligned}$$

and similarly

$$y_P' = x_P\sin\theta + y_P\cos\theta$$

so the corresponding action on the plane is just a rotation through an angle θ about the origin.

In the spherical picture we have, by an even easier calculation,

$$\begin{aligned} x_S' &= x_S\cos\theta - y_S\sin\theta \\ y_S' &= x_S\sin\theta + y_S\cos\theta \\ z_S' &= z_S \end{aligned}$$

so we have a rotation through an angle θ about the z-axis.

10 Translations

The two parameter family of translations

$$x'_P = x_P - a \qquad y'_P = y_P - b$$

preserve distances, so the corresponding Lorentz transformations, if any, must leave N invariant. Which Lorentz transformations are these?