MA 419 Assignment 6

Due 18 April 2007

1. The Camassa-Holm equation

$$u_t - u_{txx} + 3uu_x - 2u_x u_{xx} - uu_{xxx}$$

is in many ways very similar to the Korteweg de Vries equation. It has soliton solutions and infinitely many essentially different conservation laws. Prove the first two of these conservation laws, that the quantities

$$A = \int_{-\infty}^{+\infty} u\,dx$$

and

$$B = \int_{-\infty}^{+\infty} (u^2 + u_x^2)\,dx$$

are constant for reasonable solutions. You may take the word "reasonable" to mean Schwartz class in the x variable, though much weaker hypotheses would suffice.

Hint: This is best done by writing the conservation law in divergence form, *i.e.* $\partial P/\partial t + \partial Q/\partial x = 0$, as explained in lecture. P is your integrand. You need to find an appropriate Q.

2. Prove the assertion made in lecture, that

$$\tilde{u}(x_1, \ldots, x_n) = r^{2-n} u\left(\frac{x_1}{r^2}, \ldots, \frac{x_n}{r^2}\right)$$

is harmonic if and only if u is. Here, as in lecture, $r^2 = x_1^2 + \cdots + x_n^2$.

3. Suppose that f is continuous and bounded on $\mathbf{R}^2$ and that u is defined in the upper halfspace

$$H = \{(x, y, z) \colon z \geq 0\}$$

by

$$u(x, y, 0) = f(x, y)$$

and by the Poisson formula

$$u(x, y, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} K(x, y, z, \xi, \eta) f(\xi, \eta)\,d\xi\,d\eta$$

for $z > 0$, where

$$K(x, y, z, \xi, \eta) = \frac{z}{((x-\xi)^2 + (y-\eta)^2 + z^2)^{3/2}}.$$

Prove that u is harmonic in the interior of H.

Hint: The most straightforward way to do this is to differentiate under the integral sign. In principle you should check that this is justified, *i.e.* that the resulting integrals converge. There are less painful ways, but these require an understanding of where the given K comes from. It is probably easier to apply brute force than to try to be clever.

4. With u and f as in the previous problem, prove that u is bounded in H.

   Hint: u is given by a convolution. Apply Young's Inequality.

5. With u and f as in the previous two problems, prove that u is continuous on H.

   Hint: I gave three different arguments for the Poisson Formula for the unit disc in the plane. Each of these has analogues for the upper half space in $\mathbf{R}^3$, but some are easier than others. What I would recommend is to make the substitution

   $$s = \frac{\xi - x}{z} \qquad t = \frac{\eta - y}{z}$$

   in the integral, and then to use Lebesgue Dominated Convergence.