MA 419 Assignment 5

Due 28 February 2007

1. Prove that
$$u = \begin{cases} \frac{1}{2c} & \text{if } -ct \le x \le ct \\ 0 & \text{otherwise} \end{cases}$$
is a distribution solution to the inhomogeneous wave equation
$$u_{tt} - c^2 u_{xx} = \delta.$$
Hint: After consulting the definitions, you should see that this is a statement about the value of a certain integral. In order to prove that statement it is convenient to switch to characteristic coordinates.

2. Solve the homogeneous Laplace equation
$$u_{xx} + u_{yy} = 0$$
in the upper half plane with Dirchlet boundary conditions
$$u(x,0) = \frac{x^2 - 1}{x^2 + 1}.$$

3. Find an integral formula for the solution of the *inhomogenous* Laplace equation
$$u_{xx} + u_{yy} = f$$
in the half plane with Dirichlet boundary conditions
$$u(x,0) = \varphi(x).$$
Hint: The equation is linear, so it suffices to solve the problems
$$u_{xx} + u_{yy} = 0 \quad u(x,0) = \varphi(x)$$
and
$$u_{xx} + u_{yy} = f \quad u(x,0) = 0$$
and then add the solutions. The former problem was solved in lecture. The latter problem was not. I solved $u_{xx} + u_{yy} = f$ in the whole plane without boundary conditions. To get the solution in the half plane with boundary conditions you can use the Method of Reflection.

4. Prove that there are constants $C_{j,k}$ such that if u is harmonic in the open disc of radius R about the point (ξ, η) and

$$|u(x, y)| \leq K$$

for all (x, y) there, then

$$\left| \frac{\partial^{j+k} u}{\partial x^j \partial y^k}(\xi, \eta) \right| \leq C_{j,k} K R^{-j-k}$$

You don't need to find explicit $C_{j,k}$'s and certainly shouldn't worry about finding the best possible constants. *Hint:* The case $j = k = 0$ is just the Maximum Principle. The case $j + k = 1$ was done in lecture. The method used there works in the general case as well, only the calculations are messier. Try to avoid calculating more than you actually need to.

5. A function u in the plane is said to be of polynomial growth if there are constants A, ρ and N such that

$$x^2 + y^2 \geq \rho^2$$

implies

$$|u(x, y)| \leq A(x^2 + y^2)^{N/2}.$$

Assuming the result of the previous problem, even if you didn't manage to prove it, prove that every harmonic function of polynomial growth is a polynomial. *Hint:* It follows from Taylor's theorem that a function is a polynomial if and only if all but finitely many of its partial derivatives are identically zero. Apply the result of the preceding problem to discs of large radius about an arbitrary point.