MA 419 Assignment 4

Due 31 January 2007

1. Prove that any bounded weak solution of the Wave Equation
$$u_{tt} - c^2 u_{xx} = 0$$
is a solution in the sense of distributions. Is it true that any weak solution is a distribution solution?

2. Consider the function
$$u(x, y) = \frac{1}{4\pi} \log(x^2 + y^2)$$
as a distribution in the usual way, *i.e.*
$$\langle u, \varphi \rangle = \int_{\mathbf{R}^2} u(x, y)\varphi(x, y)\, dx\, dy.$$
Prove that
$$u_{xx} + u_{yy} = \delta$$
where the derivatives are to be interpreted in the sense of distributions and δ is the Dirac distribution
$$\langle \delta, \varphi \rangle = \varphi(0, 0).$$

3. Prove, by evaluating the integral, that the Poisson Formula
$$u(r, \theta) = \frac{1}{2\pi} \int \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - \varphi) + r^2} f(\varphi)\, d\varphi$$
gives the correct answer when f is constant.

4. Suppose (x_P, y_P) and (x'_P, y'_P) are points in the unit disc $x^2 + y^2 < 1$ and that the four points $(-1, 0)$, (x_P, y_P), (x'_P, y'_P) and $(1, 0)$ lie on a common circle. Prove that there is a symmetry of the Laplace equation which leaves the disc, the circle and the points $(-1, 0)$ and $(1, 0)$ invariant, while taking (x_P, y_P) to (x'_P, y'_P).

5. Let u be the solution to the Dirichlet problem
$$u_{xx} + u_{yy} = 0$$
$$u(\cos\theta, \sin\theta) = \begin{cases} -1 & \text{if } -\pi < \theta < 0 \\ +1 & \text{if } 0 < \theta < \pi \end{cases}$$
Prove that u is constant on each circle passing through the points $(-1, 0)$ and $(1, 0)$. You may use the result of the preceding problem, even if you didn't succeed in proving it.