UNIVERSITY OF DUBLIN
TRINITY COLLEGE

Faculty of Science

school of mathematics

Scholarship Exam 2007

Course 216

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ATTEMPT FOUR QUESTIONS

Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

1. *(25 points)* For each of the following systems, determine whether *all* solutions satisfy

$$\lim_{t\to+\infty} x(t) = 0 = \lim_{t\to+\infty} y(t).$$

A complete proof is not required, but you must indicate how your answers were obtained in order to receive credit.

(a) *(5 points)*

$$x'(t) = -3x(t) + 2y(t)$$
$$y'(t) = -8x(t) + 5y(t)$$

(b) *(5 points)*

$$x'(t) = 3x(t) - 2y(t)$$
$$y'(t) = 8x(t) - 5y(t)$$

(c) *(5 points)*

$$x'(t) = -5x(t) - 8y(t)$$
$$y'(t) = 3x(t) + 5y(t)$$

(d) *(5 points)*

$$x'(t) = 8x(t) + 10y(t)$$
$$y'(t) = -5x(t) - 6y(t)$$

(e) *(5 points)*

$$x'(t) = -8x(t) - 10y(t)$$
$$y'(t) = 5x(t) + 6y(t)$$

2. *(25 points)* Given that

$$\begin{pmatrix} -22 & 15 \\ -39 & 26 \end{pmatrix} = \begin{pmatrix} 2+i & 2-i \\ 3+2i & 3-2i \end{pmatrix} \begin{pmatrix} 2+3i & 0 \\ 0 & 2-3i \end{pmatrix} \begin{pmatrix} (2+3i)/2 & (-1-2i)/2 \\ (2-3i)/2 & (-1+2i)/2 \end{pmatrix},$$

compute

$$\exp\left(\begin{pmatrix} -22 & 15 \\ -39 & 26 \end{pmatrix} t\right).$$

3. Verify that

$$x_1(t) = t^3 - t$$

is a solution of the second order linear differential equation

$$(t^2 - 1)^2 x''(t) - 4t(t^2 - 1)x'(t) + (6t^2 + 2)x(t) = 0$$

in the interval $(-1, 1)$ and find a basis for the space of solutions.

4. *(25 points)*

   (a) *(4 points)* Prove that

   $$V(x, y) = \cosh(x) + y^2$$

   is an invariant of the autonomous system

   $$x'(t) = 2x(t)y(t)^2$$

   $$y'(t) = -x(t)y(t)\sinh(x(t))$$

   (b) *(7 points)* Prove that all solutions are bounded.

   (c) *(4 points)* Prove that $(0, 0)$ is an equilibrium.

   (d) *(10 points)* Prove that this equilibrium is stable but not strictly stable.

5. *(25 points)*

   (a) *(10 points)* Find a basis for the space of solutions to the differential equation

   $$x'''(t) + 2x''(t) + x'(t) = 0.$$

   (b) *(15 points)* Prove that if x_1 and x_2 are twice continuously differentiable functions defined on an interval and

   $$x_1(t)x_2'(t) - x_2 x_1'(t) \neq 0$$

   throughout this interval then there is a second order linear differential equation such that x_1 and x_2 form a basis for the space of solutions.