MA 216 Assignment 4

Due 7 February 2007

Solutions

1. The Laguerre Differential Equation is

$$tx''(t) + (1-t)x'(t) + \nu x(t) = 0.$$

For which ν does this equation have a polynomial solution? For these ν, what is this solution?
Solution: We look for solutions of the form

$$x(t) = \sum_{n=0}^{\infty} a_n t^n.$$

Substituting into the differential equation and differentiating term by term, which is always legitimate for convergent power series, we find

$$\sum_{n=0}^{\infty} n^2 a_n t^{n-1} - \sum_{n=0}^{\infty} (n-\nu) a_n t^n = 0$$

and, after changing the variable of summation,

$$\sum_{m=0}^{\infty} (m+1)^2 a_{m+1} t^m - \sum_{m=0}^{\infty} (m-\nu) a_m t^m = 0$$

This equation holds if and only if

$$(m+1)^2 a_{m+1} = (m-\nu) a_m$$

for all $m \geq 0$.

For a polynomial solution we need that $a_n \neq 0$ while $a_{n+1} = 0$ for some n, which is then the degree of the polynomial. This happens exactly when $n = \nu$. There is therefore a nonzero polynomial solution if and only if ν is a non-negative integer

Whether or not ν is an integer, an easy induction shows that

$$a_n = \frac{(n-\nu-1)(n-\nu-2)\cdots(-\nu)}{(n!)^2} a_0.$$

Our polynomial solutions are therefore

$$x(t) = a_0 \sum_{n=0}^{\nu} \frac{(n-\nu-1)(n-\nu-2)\cdots(-\nu)}{n!^2} t^n.$$

An alternative is to start the induction at the other end,

$$a_{\nu-k} = (-1)^{\nu-k} \frac{(\nu-k+1)^2(\nu-k+2)^2 \cdots (\nu)^2}{k!} a_\nu = (-1)^{\nu-k} \frac{\nu!^2}{k!(\nu-k)!^2}$$

and the solutions are

$$x(t) = a_\nu \sum_{k=0}^{\nu} (-1)^{\nu-k} \frac{\nu!^2}{k!(\nu-k)!^2} t^{\nu-k}$$

2. Suppose g is a continuously differentiable function with $g(0) \neq 0$. 0 is an equilibrium of
$$x'(t) = x(t)g(x(t)).$$
Under what conditions on g is this equilibrium stable? Under what conditions is it strictly stable?
Solution: From looking at the linearisation
$$\xi'(t) = g(0)\xi(t)$$
we see that the equilibrium is strictly stable if $g(0) < 0$ and unstable if $g(0) > 0$.

3. Find the equilibria of the system
$$x'(t) = x(t)^2 + 3y(t)^2 - 4,$$
$$y'(t) = 8x(t)^2 + y(t)^2 - 9.$$
Which are stable? Which are strictly stable?
Solution: The equilibria occur where
$$x^2 + 3y^2 - 4 = 0 = 8x^2 + y^2 - 9$$
which happens for
$$(x, y) = (\pm 1, \pm 1).$$
The coefficient matrix in the linearised problem is
$$\begin{pmatrix} 2x & 6y \\ 16x & 2y \end{pmatrix}.$$
The trace is $2x+2y$ and the determinant is $-92xy$. If $(x, y) = (-1, -1)$ then we have two complex roots with negative real part, so the equilibrium is strictly stable. In all other cases there is at least one root with positive real part, so the equilibrium is unstable.

4. Prove that

$$V(x,y) = 5x^2 + 8xy + 5y^2$$

is a Lyapunov function, but not a strict Lyapunov function for the system

$$\begin{aligned} x' &= -15x^3 - 24x^2y - 15xy^2 - 4x - 5y, \\ y' &= -15x^2y - 24xy^2 - 15y^3 + 5x + 4y. \end{aligned}$$

Solution: We see immediately that V is continuously differentiable and $V(0,0) = 0$. From

$$V(x,y) = \frac{9}{2}(x+y)^2 + \frac{1}{2}(x-y)^2$$

we see that $V(x,y) > 0$ if $(x,y) \neq 0$. From

$$\begin{aligned} \frac{\partial V}{\partial x}x' + \frac{\partial V}{\partial y}y' = & (10x + 8y)(-15x^3 - 24x^2y - 15xy^2 - 4x - 5y) \\ & + (8x + 10y)(-15x^2y - 24xy^2 - 15y^3 + 5x + 4y) \\ = & -150x^4 - 480x^3y - 684x^2y^2 - 480xy^3 - 150y^4 \\ = & -6V(x,y)^2 \leq 0 \end{aligned}$$

it follows V is a Lyapunov function. If it were a strict Lyapunov function then there would be a positive constant c such that

$$-6V(x,y)^2 < -cV(x,y)$$

for all (x,y) near $(0,0)$. But then

$$V(x,y) > \frac{c}{6}$$

for all such (x,y). There is, however, no such c, because

$$\lim_{(x,y)\to(0,0)} V(x,y) = 0.$$