MA 216 Assignment 3

Due 17.00, 8 December 2006 in the Maths Office

Solutions

1. Find all solutions of the differential equation

$$x'(t) + tx(t) + t^2 = 0.$$

Solution: This is a first order linear equation. Luckily it is a scalar equation, so we can solve it more or less explicitly.

The corresponding homogeneous equation would be

$$x'(t) + tx(t) = 0$$

or

$$\frac{x'(t)}{x(t)} = -t.$$

The left hand side is the derivative of $\log(x(t))$, so, by the Fundamental Theorem of Calculus,

$$\log(x(t)) - \log(x(0)) = \int_0^t (-s)\, ds = -\frac{t^2}{2}$$

and hence

$$x(t) = x(0) \exp\left(-\frac{t^2}{2}\right).$$

The preceding calculation gives the solution to the homogeneous equation $x'(t) + tx(t) = 0$ rather than the inhomogeneous equation $x'(t) + tx(t) + t^2 = 0$ which you were asked to solve. While it doesn't solve the problem, it does suggest that we should consider the quantitity

$$z(t) = x(t) \exp\left(\frac{t^2}{2}\right).$$

For the homogeneous equation z is an invariant. For the inhomogeneous equation it satisfies the equation

$$z'(t) = x'(t) \exp\left(\frac{t^2}{2}\right) + tx(t) \exp\left(\frac{t^2}{2}\right) = -t^2 \exp\left(\frac{t^2}{2}\right).$$

Applying the Fundamental Theorem of Calculus again,

$$z(t) - z(0) = -\int_0^t s^2 \exp\left(\frac{s^2}{2}\right) ds$$

and hence

$$x(t) = x(0) \exp\left(-\frac{t^2}{2}\right) - \exp\left(-\frac{t^2}{2}\right) \int_0^t s^2 \exp\left(\frac{s^2}{2}\right) ds.$$

This is as much of an explicit solution as we are going to get, as the integral is not elementary.

2. Solve the initial value problem

$$\begin{aligned} x'(t) &= x(t) + \exp(t)y(t), \\ y'(t) &= \qquad\qquad\quad -y(t), \end{aligned}$$

$$\begin{aligned} x(0) &= 0, \\ y(0) &= 1. \end{aligned}$$

Solution: This is an upper triangular system, and thus can be solved by solving the equations in reverse order.

The last equation is easily solved,

$$y(t) = y(0) \exp(-t) = \exp(-t).$$

Substituting into the first equation,

$$x'(t) = x(t) + 1.$$

This can be solved in a variety of ways. The simplest is to differentiate to get the second order inhomogeneous equation

$$x''(t) - x'(t) = 0$$

whose general solution is

$$x(t) = a \exp(t) + b.$$

Substituting back into the equation $x'(t) = x(t)+1$, we see that $b = -1$. The initial condition $x(0) = 0$ then shows that $a = 1$. The solution to the initial value problem is

$$\begin{aligned} x(t) &= \exp(t) - 1, \\ y(t) &= \quad \exp(-t). \end{aligned}$$

3. A particular solution to the differential equation

$$(t^4 + t^2 + 4)x''(t) - (4t^3 + 2t)x'(t) + (6t^2 + 8)x(t) = 0$$

is

$$x_1(t) = t^3 - 4t.$$

Find the general solution.
Solution: We use Wronskian reduction of order. We first solve the first order linear homogeneous equation

$$(t^4 + t^2 + 4)w'(t) - (4t^3 + 2t)w(t) = 0$$

to get

$$w(t) = \frac{w(0)}{4}(t^4 + t^2 + 4)$$

and then solve the first order linear inhomogeneous equation

$$x_1(t)x_2'(t) - x_1'(t)x_2(t) = w(t)$$

or

$$\frac{d}{dt}\frac{x_2(t)}{x_1(t)} = \frac{w(t)}{x_1(t)^2}$$

for x_2. Substituting the given expression for x_1 and the expression just found for w,

$$\frac{d}{dt}\frac{x_2(t)}{x_1(t)} = \frac{w(0)}{4}\frac{t^4 + t^2 + 4}{(t^3 - 4t)^2}.$$

The partial fractions expansion of the rational function on the right is

$$\frac{t^4 + t^2 + 4}{(t^3 - 4t)^2} = \frac{3}{8}\frac{1}{(t+2)^2} + \frac{1}{4}\frac{1}{t^2} + \frac{3}{8}\frac{1}{(t-2)^2}$$

so

$$\frac{x_2(t)}{x_1(t)} = \frac{x_2(0)}{x_1(0)} - \frac{w(0)}{4}\left(\frac{3}{8}\frac{1}{t-2} + \frac{1}{4}\frac{1}{t} + \frac{3}{8}\frac{1}{t+2}\right) = \frac{x_2(0)}{x_1(0)} - \frac{w(0)}{4}\frac{t^2 - 1}{t^3 - 4t}.$$

Substituting the given expression for $x_1(t)$ and

$$x_1(0)x_2'(0) - x_1'(0)x_2(0) = 4x_2(0)$$

for $w(0)$ and doing a bit of algebra gives

$$x_2(t) = -x_2(0)(t^2 - 1) - \frac{x_2'(0)}{4}(t^3 - 4t).$$

4. The differential equation

$$x''(t) + (2n + 1 - t^2)x(t) = 0,$$

which appears in the study of the harmonic oscillator in Quantum Mechanics, has a solution of the form

$$x(t) = H_n(t) \exp(-t^2/2),$$

where H_n is a polynomial of degree n. These are called the Hermite Polynomials, but neither the name nor their exact form is relevant to this problem. Show that

$$\int_{-\infty}^{+\infty} H_m(t)H_n(t) \exp(-t^2)\, dt = 0$$

if $m \neq n$.

Solution: This is very similar to the argument given in class for the Legendre Polynomials. Define

$$x_n(t) = H_n(t) \exp(-t^2/2)$$

Consider the integral

$$I_{m,n}(u, v) = \int_u^v \left(x'_m(t)x'_n(t) + t^2 x_m(t)x_n(t) \right) dt.$$

By the quotient rule,

$$\frac{d}{dt} x'_m(t)x_n(t) = x''_m(t)x_n(t) + x'_m(t)x'_n(t)$$

and hence, by the Fundamental Theorem of Calculus

$$x'_m(v)x_n(v) - x'_m(u)x_n(u) = \int_u^v x''_m(t)x_n(t)\, dt + \int_u^v x'_m(t)x'_n(t)\, dt.$$

From this we see that

$$I_{m,n}(u, v) = x'_m(v)x_n(v) - x'_m(u)x_n(u) - \int_u^v \left(x''_m(t) - t^2 x_m(t) \right) x_n(t)\, dt$$

or, using the differential equation for x_m,

$$I_{m,n}(u, v) = x'_m(v)x_n(v) - x'_m(u)x_n(u) + (2m + 1) \int_u^v x_m x_n(t)\, dt$$

Because of the exponential factor,

$$\lim_{v \to +\infty} x'_m(v)x_n(v) = 0 = \lim_{u \to -\infty} x'_m(u)x_n(u)$$

and hence

$$\lim_{v\to+\infty}\lim_{u\to-\infty} I_{m,n}(u,v) = (2m+1)\int_{-\infty}^{+\infty} x_m x_n(t)\,dt.$$

From the defnition it is clear that

$$I_{m,n}(u,v) = I_{n,m}(u,v),$$

so exactly the same argument, with the roles of m and n reversed, gives

$$\lim_{v\to+\infty}\lim_{u\to-\infty} I_{m,n}(u,v) = (2n+1)\int_{-\infty}^{+\infty} x_m x_n(t)\,dt.$$

It follows that

$$(2m+1)\int_{-\infty}^{+\infty} x_m x_n(t)\,dt = (2n+1)\int_{-\infty}^{+\infty} x_m x_n(t)\,dt$$

and hence that

$$\int_{-\infty}^{+\infty} x_m x_n(t)\,dt = 0$$

unless $m = n$.