MA 216 Assignment 2

Due 15 November 2006

Solutions

1. Solve
$$x'(t) = 22x(t) - 49y(t),$$
$$y'(t) = 9x(t) - 20y(t).$$

Solution: The equation in matrix form is
$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 22 & -49 \\ 9 & -20 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$
The characteristic polynomial is
$$\det(\begin{pmatrix} \lambda - 22 & 49 \\ -9 & \lambda + 20 \end{pmatrix}) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2.$$
The Jordan canonical form is then
$$J = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$
It cannot be $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ because that would give $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, independently of what P is. Once we know J, the matrix P is then found by solving the system $AP = PJ$,
$$\begin{aligned} 22p_{11} - 49p_{21} &= 1p_{11} + 1p_{12} & 22p_{12} - 49p_{22} &= 0p_{11} + 1p_{12} \\ 9p_{11} - 20p_{21} &= 1p_{21} + 1p_{22} & 9p_{12} - 20p_{22} &= 0p_{21} + 1p_{22} \end{aligned}$$
A solution is
$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} 5 & 7 \\ 2 & 3 \end{pmatrix}.$$
You may well get a different solution, since the space of solutions is two dimensional, but you should get the same answer in the end. Inverting,
$$P^{-1} = \begin{pmatrix} 3 & -7 \\ -2 & 5 \end{pmatrix}.$$
We can calculate $\exp(tJ)$ as follows. Since
$$\begin{pmatrix} t & 0 \\ t & t \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} + \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}$$

we know that

$$\exp(\begin{pmatrix} t & 0 \\ t & t \end{pmatrix}) = \exp(\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix})\exp(\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}).$$

The matrix $\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}$ is nilpotent, so the exponential series stops after finitely many terms–two, in fact,

$$\exp(\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix}) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

The matrix $\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$ is diagonal, so

$$\exp(\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}) = \begin{pmatrix} \exp(t) & 0 \\ 0 & \exp(t) \end{pmatrix}.$$

Multiplying,

$$\exp(\begin{pmatrix} t & 0 \\ t & t \end{pmatrix}) = \begin{pmatrix} \exp(t) & 0 \\ t\exp(t) & \exp(t) \end{pmatrix},$$

and, since $\exp(tA) = P\exp(tJ)P^{-1}$,

$$\exp(t\begin{pmatrix} 22 & -49 \\ 9 & -20 \end{pmatrix}) = \begin{pmatrix} (1+21t)\exp(t) & -49t\exp(t) \\ 9t\exp(t) & (1-21t)\exp(t) \end{pmatrix}.$$

Then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} (1+21t)\exp(t) & -49t\exp(t) \\ 9t\exp(t) & (1-21t)\exp(t) \end{pmatrix}\begin{pmatrix} x(0) \\ y(0) \end{pmatrix}.$$

2. Solve

$$x'(t) = 11x(t) - 30y(t),$$
$$y'(t) = 4x(t) - 11y(t).$$

Solution: The calculation is very similar, to the previous one, but slightly easier. The characteristic polynomial is

$$(\lambda - 11)(\lambda + 11) - (-30)(4) = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1),$$

so

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

One of many possible P's is

$$P = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \quad P^{-1} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}.$$

Then

$$\exp(tJ) = \begin{pmatrix} \exp(t) & 0 \\ 0 & \exp(-t) \end{pmatrix}$$

and

$$\exp(tA) = \begin{pmatrix} 6\exp(t) - 5\exp(-t) & -15\exp(t) + 15\exp(-t) \\ 2\exp(t) - 2\exp(-t) & -5\exp(t) + 6\exp(-t) \end{pmatrix}$$

so

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 15\exp(t) - 14\exp(-t) & -35\exp(t) + 35\exp(-t) \\ 6\exp(t) - 6\exp(-t) & -14\exp(t) + 15\exp(-t) \end{pmatrix} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}.$$

3. Solve

$$x'(t) = 15x(t) - 8y(t),$$
$$y'(t) = 20x(t) - 9y(t).$$

Solution: This one is similar to the previous two, with the added complication that the characteristic polynomial,

$$\lambda^2 - 6\lambda + 25$$

has complex roots. The algebra is then a bit messier, but

$$J = \begin{pmatrix} 3+4i & 0 \\ 0 & 3-4i \end{pmatrix}$$

$$P = \begin{pmatrix} 1+i & 1-i \\ 2+i & 2-i \end{pmatrix}$$

and

$$P^{-1} = \begin{pmatrix} \frac{2-i}{2i} & \frac{-1+i}{2i} \\ \frac{-2-i}{2i} & \frac{1+i}{2i} \end{pmatrix}$$

Then $\exp(tJ)$ is

$$\begin{pmatrix} \exp(3t)(\cos(4t) + i\sin(4t)) & 0 \\ 0 & \exp(3t)(\cos(4t) + i\sin(4t)) \end{pmatrix},$$

and

$$\exp(tA) = \begin{pmatrix} w_{11}(t) & w_{12}(t) \\ w_{21}(t) & w_{22}(t) \end{pmatrix}$$

where

$$w_{11}(t) = \exp(3t)\cos(4t) + 3\exp(3t)\sin(4t)$$
$$w_{12}(t) = -2\exp(3t)\sin(4t)$$
$$w_{21}(t) = 5\exp(3t)\sin(4t)$$
$$w_{22}(t) = \exp(3t)\cos(4t) - 3\exp(3t)\sin(4t)$$

and

$$x(t) = w_{11}(t)x(0) + w_{12}(t)y(0)$$
$$y(t) = w_{21}(t)x(0) + w_{22}(t)y(0)$$

4. Under what conditions on a, b, c and d is it true that *all* solutions of

$$x'(t) = ax(t) + by(t)$$

$$y'(t) = cx(t) + dy(t)$$

satisfy

$$\lim_{t\to+\infty} x(t) = 0 = \lim_{t\to+\infty} y(t)?$$

Solution: The characteristic polynomial of the coefficient matrix of the corresponding first order system is

$$\lambda^2 - (a+d)\lambda + ad - bc.$$

Depending on the sign of

$$\Delta = (a+d)^2 - 4(ad - bc)$$

we have two real roots

$$\frac{a+d}{2} \pm \frac{\sqrt{\Delta}}{2}$$

if $\Delta > 0$ or two complex roots

$$\frac{a+d}{2} \pm i\frac{\sqrt{-\Delta}}{2}$$

if $\Delta < 0$, or a real root

$$\frac{a+d}{2}$$

if $\Delta = 0$.

Consider first the case $\Delta > 0$. Set

$$\lambda_1 = \frac{a+d}{2} + \frac{\sqrt{\Delta}}{2}$$

and

$$\lambda_1 = \frac{a+d}{2} - \frac{\sqrt{\Delta}}{2}.$$

The Jordan canonical form of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and its exponential is

$$\begin{pmatrix} \exp(\lambda_1 t) & 0 \\ 0 & \exp(\lambda_2 t) \end{pmatrix}.$$

From $\exp(tA) = P\exp(tJ)P^{-1}$ we obtain a solution of the form

$$x(t) = w_{11}(t)x(0) + w_{12}(t)y(0)$$

$$y(t) = w_{21}(t)x(0) + w_{22}(t)y(0)$$

where

$$w_{11}(t) = \frac{p_{11}p_{22}\exp(\lambda_1 t) - p_{12}p_{21}\exp(\lambda_2 t)}{p_{11}p_{22} - p_{12}p_{21}}$$

$$w_{12}(t) = \frac{-p_{11}p_{12}\exp(\lambda_1 t) - p_{12}p_{11}\exp(\lambda_2 t)}{p_{11}p_{22} - p_{12}p_{21}}$$

$$w_{21}(t) = \frac{p_{21}p_{22}\exp(\lambda_1 t) - p_{22}p_{21}\exp(\lambda_2 t)}{p_{11}p_{22} - p_{12}p_{21}}$$

$$w_{22}(t) = \frac{-p_{21}p_{12}\exp(\lambda_1 t) + p_{22}p_{11}\exp(\lambda_2 t)}{p_{11}p_{22} - p_{12}p_{21}}$$

The best way to think about this is as the vector equation

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} p_{11} \\ p_{21} \end{pmatrix} \exp(\lambda_1 t)\alpha + \begin{pmatrix} p_{12} \\ p_{22} \end{pmatrix} \exp(\lambda_2 t)\beta$$

where

$$\alpha = \frac{p_{22}x(0) - p_{12}y(0)}{p_{11}p_{22} - p_{12}p_{21}}$$

$$\beta = \frac{-p_{21}x(0) + p_{11}y(0)}{p_{11}p_{22} - p_{12}p_{21}}$$

In other words the set

$$\left\{ \begin{pmatrix} p_{11} \\ p_{21} \end{pmatrix} \exp(\lambda_1 t), \begin{pmatrix} p_{12} \\ p_{22} \end{pmatrix} \exp(\lambda_2 t) \right\}$$

is a basis for the vector space of solutions to the differential equation. We don't know what P is, but

$$\lim_{t \to +\infty} \exp(\lambda_1 t) = 0 = \lim_{t \to +\infty} \exp(\lambda_2 t)$$

if λ_1 and λ_2 are both negative. Conversely if λ_1 is non-negative then

$$x(t) = p_{11} \exp(\lambda_1 t)$$

$$y(t) = p_{21} \exp(\lambda_1 t)$$

is a solution and at least one of x or y does not converge to zero. Note that p_{11} and p_{21} cannot both be zero, since P would then fail to be invertible. Similarly, if λ_2 is non-negative then

$$x(t) = p_{12} \exp(\lambda_2 t)$$

$$y(t) = p_{22} \exp(\lambda_2 t)$$

is a solution and at least one of x or y does not converge to zero. Thus a necessary and sufficient condition in the case $\Delta > 0$ is that both λ_1 and λ_2 are negative.

In the case $\Delta < 0$ things are a bit messier, but the same idea can be made to work. Our Jordan matrix is

$$J = \begin{pmatrix} \mu + i\nu & 0 \\ 0 & \mu - i\nu \end{pmatrix}.$$

We may assume that the second column of P is the complex conjugate of the first, since the two columns of P are simply eigenvectors of A with eigenvalues $\mu + i\nu$ and $\mu - i\nu$ and the complex conjugate of an eigenvector with eigenvalue $\mu + i\nu$ will be an eigenvector with eigenvalue $\mu - i\nu$. Thus P can be taken to have the form

$$P = \begin{pmatrix} p + iq & p - iq \\ r + is & r - is \end{pmatrix}$$

for some p, q, r and s. Then

$$P^{-1} = \frac{i}{2}(pq - rs)^{-1} \begin{pmatrix} r - is & -p + iq \\ -r - is & p + iq \end{pmatrix}$$

and, after much algebra,

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} \alpha + \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} \beta$$

where

$$\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = \begin{pmatrix} p\exp(\mu t)\cos(\nu t) - q\exp(\mu t)\sin(\nu t) \\ r\exp(\mu t)\cos(\nu t) - s\exp(\mu t)\sin(\nu t) \end{pmatrix}$$

$$\begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} q\exp(\mu t)\cos(\nu t) + p\exp(\mu t)\sin(\nu t) \\ s\exp(\mu t)\cos(\nu t) + r\exp(\mu t)\sin(\nu t) \end{pmatrix}$$

$$\alpha = \frac{sx(0) - qy(0)}{pq - rs}$$

$$\beta = \frac{-rx(0) + py(0)}{pq - rs}$$

Regardless of what p, q, r and s are, if μ is negative then

$$\lim_{t\to+\infty} x(t) = 0 = \lim_{t\to+\infty} y(t).$$

Conversely, if $\mu \geq 0$ then either $x(t)$ or $y(t)$ fails to tend to zero for at least one of the solutions (x_1, y_1) or (x_2, y_2). Thus a necessary and sufficient condition in the case $\Delta < 0$ is that $\mu < 0$.

The simplest case is that of a repeated real root, $\Delta = 0$. Set

$$\lambda = \frac{a+d}{2}.$$

The repeated root is λ and there are two possibilities for the Jordan form.

If the Jordan form is

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

then

$$x(t) = x(0)\exp(\lambda t)$$

and

$$y(t) = y(0)\exp(\lambda t)$$

If $\lambda < 0$ then all solutions tend to zero. If $\lambda \geq 0$ then no non-trivial solutions tend to zero.

If the Jordan form is

$$\begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$$

then things are slightly more complicated. Multiplying

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} \exp(t) & 0 \\ t\exp(t) & \exp(t) \end{pmatrix} \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}^{-1} = \begin{pmatrix} (1 + \frac{a-d}{2}t)\exp(\lambda t) & bt\exp(\lambda t) \\ ct\exp(\lambda t) & (1 + \frac{d-a}{2}t)\exp(\lambda t) \end{pmatrix}$$

shows, however, that we have the same conclusion. If $\lambda < 0$ then all solutions tend to zero. If $\lambda \geq 0$ then no non-trivial solutions tend to zero.

In all cases the conclusion is the same. A necessary and sufficient condition for all solutions to tend to zero is that the roots of the characteristic polynomial have negative real parts. An equivalent condition, which is a bit easier to chaeck, is that $a + d$ and $ad - bc$ are both positive.

5. Find a basis for the vector space of solutions to

$$x'''(t) - x''(t) - x'(t) + x(t) = 0.$$

Solution: The characteristic polynomial of the coefficient matrix of the corresponding first order system is

$$\lambda^3 - \lambda^2 - \lambda + 1 = (\lambda - 1)^2(\lambda + 1)$$

The Jordan canonical form would be

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

whose exponential is

$$\exp(tJ) = \begin{pmatrix} \exp(t) & 0 & 0 \\ t\exp(t) & \exp(t) & 0 \\ 0 & 0 & \exp(-t) \end{pmatrix}.$$

Multiplying from the left by P, whatever it is, and on the right by P^{-1} will give us some linear combinations of

$$\{\exp(t), t\exp(t), \exp(-t)\},$$

which is therefore the basis we are looking for.