MA 216 Assignment 1

Due Wednesday 25 October 2006

Solutions

1. For each of the following, say whether it is a scalar equation or system, give the order, and state whether it is linear or non-linear. If linear, state whether it is homogeneous or inhomogenous.

   (a) Van der Pol's Equation:
   $$x''(t) + (1 - x(t)^2)x'(t) + x(t) = 0$$

   (b) Bessel's Equation:
   $$t^2x''(t) + tx'(t) + (t^2 - \nu^2)x(t) = 0$$

   (c) The Emden-Fowler Equation
   $$tx''(t) + 2x'(t) + at^\nu x(t)^n = 0.$$

   (d) The Lotka-Volterra Model:
   $$x'(t) = x(t)(\alpha - \beta y(t)) \qquad y'(t) = -y(t)(\gamma - \delta x(t))$$

   Solution:

   (a) Van der Pol's Equation is a second order nonlinear scalar equation.

   (b) Bessel's Equation: is a second order linear homogeneous scalar equation.

   (c) The Emden-Fowler Equation is a second order nonlinear scalar equation, unless $n = 1$, in which case it is linear homogeneous, or $n = 0$, in which case it is linear inhomogeneous.

   (d) The Lotka-Volterra Model is a first order nonlinear system.

2. Which of the equations (or systems) from the preceding exercise have translation symmetry in the independent variable (t as the equations are written above)?
   Solution: Only Van der Pol's equation and the Lotka-Volterra system have translation symmetry.

3. (a) Prove that

$$z(t) = x(t)y(t)$$

is an invariant of the system

$$x'(t) = x(t) \qquad y'(t) = -y(t).$$

Note: although the system is easy to solve, there is no need to do so.

(b) Prove that

$$x'(t)^2 + x(t)^4$$

is an invariant of

$$x''(t) + 2x(t)^3 = 0$$

and use this fact to prove that all solutions of the equation are bounded.

Solution:

(a) The familiar properties of the derivative, *e.g.* the sum and product rule, show that

$$z'(t) = x'(t)y(t) + x(t)y'(t) = x(t)y(t) - x(t)y(t) = 0,$$

so z is constant.

(b) Similarly, setting

$$w(t) = x'(t)^2 + x(t)^4,$$

one sees that

$$w'(t) = 2x'(t)x''(t) + 4x(t)^3x'(t) = 2x'(t)(x''(t) + 2x(t)^3) = 0,$$

so w is constant. It follows, since $x'(t)^2$ is non-negative, that

$$-\sqrt[4]{w(0)} \le x(t) \le \sqrt[4]{w(0)}.$$

4. Prove that

$$\|AB\|_\infty \le \|A\|_\infty \|B\|_\infty.$$

Solution: Set

$$C = AB.$$

Then, by definition,

$$c_{il} = \sum_{j=1}^{k} a_{ij}b_{jl}$$

and, by the triangle inequality,

$$|c_{il}| \leq \sum_{j=1}^{k} |a_{ij}||b_{jl}|$$

and

$$\sum_{l=1}^{k} |c_{il}| \leq \sum_{l=1}^{k} \sum_{j=1}^{k} |a_{ij}||b_{jl}|.$$

The sums being finite, the order of summation is unimportant,

$$\sum_{l=1}^{k} |c_{il}| \leq \sum_{j=1}^{k} \sum_{l=1}^{k} |a_{ij}||b_{jl}|.$$

In the inner sum, the factor $|a_{ij}|$ is independent of l and may be pulled outside the sum,

$$\sum_{l=1}^{k} |c_{il}| \leq \sum_{j=1}^{k} |a_{ij}| \sum_{l=1}^{k} |b_{jl}|.$$

For any j the definition of $\|B\|_\infty$ shows that

$$\sum_{l=1}^{k} |b_{jl}| \leq \|B\|_\infty$$

and hence

$$\sum_{l=1}^{k} |c_{il}| \leq \sum_{j=1}^{k} |a_{ij}|\|B\|_\infty.$$

The factor $\|B\|_\infty$ is independent of j, and hence may be pulled outside the sum,

$$\sum_{l=1}^{k} |c_{il}| \leq \|B\|_\infty \sum_{j=1}^{k} |a_{ij}| \leq \|B\|_\infty \|A\|_\infty.$$

This is true for all i, so

$$\|C\|_\infty = \max_{1 \leq i \leq k} \sum_{l=1}^{k} |c_{il}| \leq \|B\|_\infty \|A\|_\infty.$$

5. A vector valued function x is said to be continuous at t if, for all positive ϵ, there is a positive δ, such that

$$|s - t| < \delta$$

implies

$$\|x(s) - x(t)\| < \epsilon.$$

Show that it doesn't matter which norm we take, $\|x\|_1$ or $\|x\|_\infty$, in this definition, *i.e.* that the same set of functions are continuous regardless of which norm is used.

Solution: First, note that for all vectors v in $\mathbf{R}^k$,

$$\|v\|_\infty \leq \|v\|_1$$

and

$$\|v\|_1 \leq k\|v\|_\infty.$$

This follows immediately from the definitions.

Suppose that x is continuous, continuity being defined in terms of the 1-norm. For any positive ϵ there is then a positive δ such that

$$|s - t| < \delta$$

implies

$$\|x(s) - x(t)\|_1 < \epsilon.$$

Then, since $\|x(s) - x(t)\|_\infty \leq \|x(s) - x(t)\|_1$,

$$|s - t| < \delta$$

implies

$$\|x(s) - x(t)\|_\infty < \epsilon.$$

Thus x is continuous, continuity being defined in terms of the ∞-norm.

Suppose now that x is continuous, continuity being defined in terms of the ∞-norm. For any positive θ there is then a positive δ such that For any positive ϵ there is then a positive δ such that

$$|s - t| < \delta$$

implies

$$\|x(s) - x(t)\|_\infty < \theta.$$

In particular, for any positive ϵ one has $\frac{\epsilon}{k} > 0$ and hence there is a positive δ such that

$$|s - t| < \delta$$

implies

$$\|x(s) - x(t)\|_\infty < \frac{\epsilon}{k}.$$

Then, since $\|x(s) - x(t)\|_1 \leq k\|x(s) - x(t)\|_\infty$,

$$|s - t| < \delta$$

implies

$$\|x(s) - x(t)\|_1 < k\frac{\epsilon}{k} = \epsilon.$$

Thus x is continuous, continuity being defined in terms of the 1-norm.