UNIVERSITY OF DUBLIN
TRINITY COLLEGE

FACULTY OF SCIENCE

SCHOOL OF MATHEMATICS

M. Kolesar **Special Annual Exam 2007**

COURSE 216

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ATTEMPT FOUR QUESTIONS

Log tables are available from the invigilators, if required.

Non-programmable calculators are permitted for this examination,—please indicate the make and model of your calculator on each answer book used.

1. *(25 points)* Find all solutions of the first order linear inhomogeneous equation

$$tx'(t) - x(t) = t^2 \cos(t)$$

for $t > 0$.

2. *(25 points)* Given that

$$\begin{pmatrix} 17 & -9 \\ 25 & -13 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix},$$

solve the initial value problem

$$x'(t) = 17x(t) - 9y(t),$$

$$y'(t) = 25x(t) - 13y(t),$$

$$x(0) = y(0) = 1.$$

3. *(25 points, 5 points each)* Give examples of

   (a) a system of differential equations such that the space of solutions *is not* a vector space,

   (b) an equilibrium of an autonomous system which is stable but not asymptotically stable,

   (c) a first order scalar differential equation for which the initial value problem *does not* have a unique solution,

   (d) a differential equation for which all nonzero solutions are unbounded,

   (e) a pair of matrices A and B such that $\exp(A + B) \neq \exp(A)\exp(B)$.

4. *(25 points)*

   (a) *(3 points)* Prove that $(0, 0)$ is an equilibrium of the autonomous system

$$x'(t) = -y(t) - x(t)^5$$

$$y'(t) = x(t) - y(t)^5$$

(b) *(8 points)* Prove that

$$V(x, y) = x^2 + y^2$$

is a Lyapunov function, but not a strict Lyapunov function.

(c) *(4 points)* What conclusions can you draw about stability or asymptotic stability from the existence of a Lyapunov function?

(d) *(10 points)* Prove that *all* solutions are bounded.

5. *(25 points)*

(a) *(10 points)* Find a basis for the space of solutions to the linear homogeneous differential equation

$$x''''(t) + 2x''(t) + x(t) = 0.$$

(b) *(15 points)* Find all solutions of the linear inhomogenous differential equation

$$x''(t) + x(t) = \sin(t).$$