MA2331 Tutorial Sheet 2, Solutions[1]

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Problem Sheet 2

1. Express the following periodic functions ($l = 2\pi$) as complex Fourier series

   (a)
$$f(x) = \begin{cases} 0 & -\pi < x < -a \\ 1 & -a < x < a \\ 0 & a < x < \pi \end{cases}$$

   where $a \in (0, \pi)$ is a constant.
   Solution: $f(x) = \sum_{n \in Z} c_n e^{inx}$ with

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx \; e^{-inx} f(x) = \frac{1}{2\pi} \int_{-a}^{a} dx \; e^{-inx}$$

   so that $c_0 = a/\pi$ and

$$c_n = \frac{1}{2\pi} \left. \frac{e^{-inx}}{-in} \right|_{-a}^{a} = \frac{1}{\pi n} \frac{e^{ian} - e^{-ian}}{2i} = \frac{1}{\pi n} \sin an.$$

   (b)
$$f(x) = \frac{1}{2 - e^{ix}} + \frac{1}{2 - e^{-ix}}.$$

   Solution: Both terms can be expanded as a geometric series which is exactly the complex Fourier series:

$$f(x) = \frac{1}{2 - e^{ix}} + \frac{1}{2 - e^{-ix}} = \frac{1}{2} \left(\frac{1}{1 - \frac{1}{2} e^{ix}} + \frac{1}{1 - \frac{1}{2} e^{-ix}} \right) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{2^{|n|}} e^{inx} + \frac{1}{2}$$

2. Show that the periodic function f defined by $f(x) = |x| - \frac{1}{2}\pi$ for $-\pi < x \leq \pi$ and $f(x + 2\pi) = f(x)$ has the Fourier series expansion

$$f(x) = -\frac{4}{\pi} \sum_{n>0, \text{ odd}} \frac{\cos nx}{n^2}.$$

   Solution: f is even so $b_n = 0$ for all n.

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} dx \; \cos nx \; \left(|x| - \frac{1}{2}\pi \right).$$

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A quick calculation gives $a_0 = 0$. For $n > 0$ use the fact that $\cos nx$ integrates to zero over a full period

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} dx\ |x|\ \cos nx = \frac{2}{\pi}\int_{0}^{\pi} dx\ x\cos nx \\
&= \frac{2}{\pi}\left(\frac{x\sin nx}{n}\bigg|_0^{\pi} - \int_0^{\pi} dx\ \frac{\sin nx}{n}\right) \\
&= 0 + \frac{2}{\pi}\frac{\cos nx}{n^2}\bigg|_0^{\pi} = \frac{2}{\pi}\frac{((-1)^n - 1)}{n^2}.
\end{aligned}$$

Thus $a_n = 0$ if n is even and $a_n = -4/(\pi n^2)$ if n odd.

3. Use the Fourier series given in question 2 to compute the following sums

$$S_1 = 1 - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} - \frac{1}{13^2} + ...$$

$$S_2 = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + ...$$

Remark: With calculations of this kind it makes sense to try a quick numerical check of your answer.

*Solution:*To compute S_1 set $x = \pi/4$ in the Fourier series quoted in question 1

$$f\left(\frac{\pi}{4}\right) = -\frac{4}{\pi}\frac{1}{\sqrt{2}}S_1.$$

Since $f(\frac{\pi}{4}) = -\frac{\pi}{4}$ one has

$$-\frac{\pi}{4} = -\frac{1}{\sqrt{2}}\frac{4}{\pi}S_1,$$

so that

$$S_1 = \frac{\sqrt{2}\pi^2}{16}.$$

ii) The average value of $|f|^2$ is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} dx\ |f(x)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} dx\ \left(|x| - \frac{1}{2}\pi\right)^2 = \frac{1}{\pi}\int_0^{\pi} dx\ \left(x - \frac{1}{2}\pi\right)^2.$$

A short calculation gives that this is equal to $\pi^2/12$. Applying Parseval's theorem

$$\frac{\pi^2}{12} = \frac{1}{4}|a_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(|a_n|^2 + |b_n|^2\right) = \frac{1}{2}\cdot\frac{16}{\pi^2}S_2,$$

and so

$$S_2 = \frac{\pi^4}{96}.$$

4. Compute the Fourier transform of $f(x) = e^{-a|x|}$ where a is a positive constant. Use the result to show that

$$\int_{-\infty}^{\infty} dp \, \frac{\cos p}{1+p^2} = \frac{\pi}{e}.$$

Solution:

$$\begin{aligned}
\tilde{f}(k) &= \int_{-\infty}^{\infty} dx \; e^{-ikx} \; f(x) \\
&= \int_{-\infty}^{\infty} dx \; e^{-ikx} \; e^{-a|x|} = \left[\int_{0}^{\infty} dx \; e^{-ikx-ax} + \int_{-\infty}^{0} dx \; e^{-ikx+ax} \right] \\
&= -\frac{e^{-x(a+ik)}}{a+ik}\Bigg|_{0}^{\infty} - \frac{e^{x(a-ik)}}{a-ik}\Bigg|_{-\infty}^{0} \\
&= \frac{1}{a+ik} + \frac{1}{a-ik} = \frac{2a}{a^2+k^2}.
\end{aligned}$$

f can be represented as a Fourier integral

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \; e^{ikx} \; \tilde{f}(k) = \frac{a}{\pi} \int_{-\infty}^{\infty} dk \; \frac{e^{ikx}}{a^2+k^2}.$$

Setting $a = 1$ and $x = 1$ gives

$$e^{-1} = \frac{1}{\pi} \int_{-\infty}^{\infty} dk \; \frac{e^{ik}}{1+k^2}.$$

Taking the real part (and multiplying by π)

$$\frac{\pi}{e} = \int_{-\infty}^{\infty} dk \; \frac{\cos k}{1+k^2}.$$