MA2331 Solutions Tutorial Sheet 1[1]

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Problem Sheet 1

1. Find the Fourier series representation of the sawtooth function f defined by $f(x) = x$ for $-\pi < x < \pi$ and $f(x + 2\pi) = f(x)$.

   Solution: f is odd so $a_n = 0$ for all n.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} dx\ x\ \sin nx = -\left.\frac{x\cos nx}{n\pi}\right|_{\pi}^{-\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos nx}{n}.$$

   The integral on the RHS is zero since it is just a cosine integrated over a full period (or n periods). Thus $b_n = -2\cos(n\pi)/n = -2(-1)^n/n$ which gives

$$f(x) = -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx.$$

2. Establish that

$$\int_{-\pi}^{\pi} dx\ \sin mx\ \sin nx = \int_{-\pi}^{\pi} dx\ \cos mx\ \cos nx = 0,$$

   if $m \neq n$ (both m and n are integers).

   Solution: In this question m and n will be taken as positive integers. The problem can be tackled using complex exponentials or trig identities. Using the identity

$$2\sin A\sin B = \cos(A - B) - \cos(A + B),$$

$$\int_{-\pi}^{\pi} dx\ \sin mx\sin nx = \frac{1}{2}\int_{-\pi}^{\pi} dx\ \left[\cos(m-n)x - \cos(m+n)x\right],$$

   which is zero (integral of cosine over full periods) provided $m - n$ and $m + n$ are non-zero. To show that

$$\int_{-\pi}^{\pi} dx\ \cos mx\cos nx = 0,$$

   use

$$2\cos A\cos B = \cos(A + B) + \cos(A - B).$$

3. The periodic function f is defined by

$$f(x) = \begin{cases} \sin x & 0 < x < \pi \\ 0 & -\pi < x < 0 \end{cases}$$

   and $f(x + 2\pi) = f(x)$.

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(a) Represent $f(x)$ as a Fourier series.

Solution: This function is neither odd nor even, though the only non-zero b_n coefficient is $b_1 = \frac{1}{2}$ (since $f(x) = \frac{1}{2}(\sin x + |\sin x|)$ and $|\sin x|$ is even). Now to the a_n coefficients

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} dx \;\; \cos nx \;\; f(x) = \frac{1}{\pi} \int_{0}^{\pi} dx \;\; \cos nx \;\; \sin x$$

This can be computed via complex exponentials or through the identity
$2 \sin A \cos B = \sin(A + B) + \sin(A - B)$:

$$a_n = \frac{1}{2\pi} \int_0^{\pi} dx \;\; [\sin(1+n)x + \sin(1-n)x] = -\frac{1}{2\pi} \left(\frac{\cos(1+n)x}{1+n} + \frac{\cos(1-n)x}{1-n} \right) \Bigg|_0^{\pi}.$$

Now $\cos(1+n)\pi = \cos(1-n)\pi = -(-1)^n$, and so

$$a_n = -\frac{1}{2\pi}(-(-1)^n - 1) \left(\frac{1}{1+n} + \frac{1}{1+n} \right) = \frac{1}{\pi}(1 + (-1)^n) \frac{1}{1-n^2}.$$

This is ambiguous for $n = 1$; it is trivial to check that $a_1 = 0$. Putting everything together

$$f(x) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{n>0,\text{even}} \frac{\cos nx}{1-n^2} + \frac{1}{2} \sin x,$$

or

$$f(x) = \frac{1}{\pi} + \frac{2}{\pi} \sum_{m=1}^{\infty} \frac{\cos 2mx}{1-4m^2} + \frac{1}{2} \sin x.$$

(b) Derive the remarkable formula

$$\frac{1}{2^2-1} + \frac{1}{4^2-1} + \frac{1}{6^2-1} + ... = \frac{1}{2}.$$

Solution: $f(0) = 0$ leads to the amazing formula

$$\frac{1}{2^2-1} + \frac{1}{4^2-1} + \frac{1}{6^2-1} + ... = \frac{1}{2}.$$