

MA2331 Tutorial Sheet 2.¹

14 October 2014
(due on October 21 in class)

Useful facts:

- A function $f(x)$ has period l if $f(x+l) = f(x)$, it is odd if $f(-x) = -f(x)$ and even if $f(-x) = f(x)$.
- A function with period l has the Fourier series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nx}{l}\right).$$

where

$$\begin{aligned} a_n &= \frac{2}{l} \int_{-l/2}^{l/2} f(x) \cos\left(\frac{2\pi nx}{l}\right) dx, \quad (n = 0, 1, 2, \dots) \\ b_n &= \frac{2}{l} \int_{-l/2}^{l/2} f(x) \sin\left(\frac{2\pi nx}{l}\right) dx, \quad (n = 1, 2, \dots) \end{aligned}$$

- A function with period l has the Fourier series expansion

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp \frac{2\pi nx}{l}.$$

where

$$c_n = \frac{1}{l} \int_{-l/2}^{l/2} f(x) \exp\left(\frac{-2\pi nx}{l}\right) dx$$

- Parseval's formula:

$$\begin{aligned} \frac{1}{l} \int_{-l/2}^{l/2} dx |f(x)|^2 &= \frac{1}{4} a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \\ &= \sum_{n=-\infty}^{\infty} |c_n|^2 \end{aligned}$$

- The Fourier integral or Fourier transform:

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \tilde{f}(k) e^{ikx} \\ \tilde{f}(k) &= \int_{-\infty}^{\infty} dx f(x) e^{-ikx} \end{aligned}$$

¹Stefan Sint, sint@maths.tcd.ie, see also <http://www.maths.tcd.ie/~sint/MA2331/MA2331.html>

Questions

1. Express the following (periodically extended) functions ($l = 2\pi$) as complex Fourier series

(a)

$$f(x) = \begin{cases} 0 & -\pi < x < -a \\ 1 & -a \leq x \leq a \\ 0 & a < x \leq \pi \end{cases}$$

where $a \in (0, \pi)$ is a constant.

(b)

$$f(x) = \frac{1}{2 - e^{ix}} + \frac{1}{2 - e^{-ix}}.$$

(hint: geometric series)

(3 marks)

2. Show that the periodic function f defined by $f(x) = |x| - \frac{1}{2}\pi$ for $-\pi < x \leq \pi$ and $f(x + 2\pi) = f(x)$ has the Fourier series expansion

$$f(x) = -\frac{4}{\pi} \sum_{n>0, \text{ odd}} \frac{\cos nx}{n^2}.$$

(2 marks)

3. Use the Fourier series given in question 2 to compute the following sums

$$S_1 = 1 - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} - \frac{1}{13^2} + \dots$$

$$S_2 = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \dots$$

Do a quick numerical check of your answer!

(3 marks)

4. Compute the Fourier transform of $f(x) = e^{-a|x|}$ where a is a positive constant. Use the result to show that

$$\int_{-\infty}^{\infty} dp \frac{\cos p}{1 + p^2} = \frac{\pi}{e}.$$

(2 marks)