MA2331 Tutorial Sheet 5, Solutions.[1]

4 December 2014
(Due 12 December 2014 in class)

Questions

1. Compute the line integrals:

   (a) $\int_C \ (dx\ xy\ + \frac{1}{2}dy\ x^2\ + dz)$ where C is the line segment joining the origin and the point $(1, 1, 2)$.

   (b) $\int_C \ (dx\ yz\ + dy\ xz\ + dz\ yx^2)$ where C is the same line as in the previous part

   Solution:

   A quick way here is to note that $\mathbf{F}$ is conservative.

   $$\mathbf{F} = xy\mathbf{i} + \frac{1}{2}x^2\mathbf{j} + \mathbf{k} = \nabla\phi \tag{1}$$

   where $\phi = \frac{1}{2}x^2y + z$. Hence

   $$\int_C \mathbf{F} \cdot d\mathbf{l} = \phi(1, 1, 2) - \phi(0, 0, 0) = \frac{5}{2}. \tag{2}$$

   For the next part, use the parametrization $x(u) = u$, $y(u) = u$, $z(u) = 2u$ $(0 \le u \le 1)$.

   $$\begin{aligned} \frac{d\mathbf{r}}{du} &= \mathbf{i} + \mathbf{j} + 2\mathbf{k}, \\ \mathbf{F} \cdot \frac{d\mathbf{r}}{du} &= 2u^2 + 2u^2 + 2u^3 = 4u^2 + 2u^3 \end{aligned} \tag{3}$$

   so

   $$\int_C \mathbf{F} \cdot d\mathbf{l} = \int_0^1 du\ \left(4u^2 + 2u^3\right) = \frac{4}{3} + \frac{1}{2} = \frac{11}{6}. \tag{4}$$

2. For each of the following vector fields compute the line integral $\oint_C \mathbf{F} \cdot \mathbf{dl}$ where C is the unit circle in the xy-plane taken anti-clockwise.

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(a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$

(b) $\mathbf{F} = y\mathbf{i} - x^2y\mathbf{j}$.

Solution:

In the first part $\mathbf{F} = \nabla\frac{1}{2}(x^2 + y^2)$ so that $\mathbf{F}$ is conservative giving $\oint_C \mathbf{F}\cdot d\mathbf{l}$. In the second part parametrize curve:

$$\begin{aligned} x(u) &= \cos u \\ y(u) &= \sin u \\ z(u) &= 0 \end{aligned} \tag{5}$$

where $0 \le u \le 2\pi$ or $\mathbf{r}(u) = \cos u\mathbf{i} + \sin u\mathbf{j}$. Now

$$\frac{d\mathbf{r}(u)}{du} = -\sin u\mathbf{i} + \cos u\mathbf{j}. \tag{6}$$

and

$$\mathbf{F}\cdot\frac{d\mathbf{r}}{du} = -y\sin u - x^2y\cos u = -\sin^2 u - \cos^3 u\sin u. \tag{7}$$

Thus

$$\oint_C \mathbf{F}\cdot d\mathbf{l} = \int_0^{2\pi} du\ \left(-\sin^2 u - \cos^3 u\sin u\right) = -\pi, \tag{8}$$

since the average value of $\sin^2 u$ is $\frac{1}{2}$ and $\int_0^{2\pi} du\ \cos^3 u\sin u = 0$ by symmetry.

3. Evaluate the line integrals $\int_C \mathbf{F}\cdot d\mathbf{l}$ for

   (a) $\mathbf{F} = (x^2y, 4, 0)$ with C given by $\mathbf{r}(t) = (\exp(t), \exp(-t), 0)$ with t going from zero to one;

   (b) $\mathbf{F} = (z, x, y)$ with C given by $\mathbf{r}(t) = (\sin t, 3\sin t, \sin^2 t)$ with t going from zero to $\pi/2$.

Solution:

For the first one

$$\mathbf{r} = (\exp(t), \exp(-t), 0) \tag{9}$$

so

$$\frac{d\mathbf{r}}{dt} = (\exp(t), -\exp(-t), 0) \tag{10}$$

and, on the curve,

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = e^{2t} - 4e^{-t} \tag{11}$$

and hence

$$\int_C \mathbf{F} \cdot \mathbf{dl} = \int_0^1 \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_0^1 (e^{2t} - 4e^{-t}) dt = \frac{1}{2}e^2 + 4e^{-1} - \frac{9}{2} \tag{12}$$

For the second one

$$\mathbf{r} = (\sin t, 3 \sin t, \sin^2 t) \tag{13}$$

so

$$\frac{d\mathbf{r}}{dt} = (\cos t, 3 \cos t, 2 \sin t \cos t) \tag{14}$$

and, on the curve,

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = (7 \sin^2 t + 3 \sin t) \cos t \tag{15}$$

and hence

$$\begin{aligned} \int_C \mathbf{F} \cdot \mathbf{dl} &= \int_0^{\pi/2} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt \\ &= \int_0^{\pi/2} (7 \sin^2 t + 3 \sin t) \cos t dt \\ &= \int_0^1 (7u^2 + 3u) du = \frac{23}{6} \end{aligned} \tag{16}$$

where we have used a substitution $u = \sin t$.

4. For each of these fields determine if $\mathbf{F}$ is conservative, if it is, by integration or otherwise, find a potential: ϕ such that $\mathbf{F} = \nabla \phi$.

   (a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$

   (b) $\mathbf{F} = 3y^2\mathbf{i} + 6xy\mathbf{j}$

   (c) $\mathbf{F} = e^x \cos y\mathbf{i} - e^x \sin y\mathbf{j}$

   (d) $\mathbf{F} = (\cos y + y \cos x)\mathbf{i} + (\sin x - x \sin y)\mathbf{j}$

Solution:

So, in the first case, it is easy to see the curl is zero, having done that we want $\mathbf{F} = \nabla\phi$, hence $F_1 = \partial_x \phi$ or

$$\frac{\partial}{\partial x}\phi = x \tag{17}$$

and hence $\phi = x^2/2 + C(y, z)$, where $C(y, z)$ is an arbitrary function of y and z, substitute that back in to get

$$\frac{\partial}{\partial y}C = y \tag{18}$$

giving $\phi = x^2/2 + y^2/2 + C(z)$ where $C(z) = C$ a constant follows from $F_3 = 0$.

For the next one the curl is again zero so there is a potential,

$$\partial_x \phi = 3y^2 \tag{19}$$

so $\phi = 3y^2 x + C(y, z)$. Substituting into the y equation gives

$$\partial_y \phi = 6xy + \partial_y C(y, z) = 6xy \tag{20}$$

and hence $\partial_y C(y, z) = 0$ so $C(y, z) = C(z)$, further substituting this into $\partial_z \phi = 0$ shows $C(z) = C$ a constant and $\phi = 3y^2 x + C$.

For the next one the curl is again zero so there is a potential,

$$\partial_x \phi = e^x \cos y \tag{21}$$

so $\phi = e^x \cos y + C(y, z)$. Substituting into the y equation and z equation show that $C(y, z) = C$ a constant and $\phi = e^x \cos y + C$.

Finally the last one also has zero curl and

$$\partial_x \phi = \cos y + y \cos x \tag{22}$$

giving $\phi = x \cos y + y \sin x + C(y, z)$ and, again, substituting in to the y equation and z equation show that $C(y, z) = C$ a constant and $\phi = x \cos y + y \sin x + C$. It won't always work out like this with arbitrary function turning out to be an arbitrary constant, it is just an accident that I ask you three examples like this!

5. Consider the 'point vortex' vector field

$$\mathbf{F} = \frac{y}{x^2+y^2}\mathbf{i} - \frac{x}{x^2+y^2}\mathbf{j}.$$

Show that curl $\mathbf{F} = 0$ away from the z-axis. Establish that $\mathbf{F}$ is *not* conservative in the (non simply-connected) domain $x^2+y^2 \geq \frac{1}{2}$. Is $\mathbf{F}$ conservative in the domain defined by $x^2+y^2 \geq \frac{1}{2}$, $y \geq 0$? If so obtain a scalar potential for $\mathbf{F}$.

Solution:

$$\begin{aligned}
\nabla \times \mathbf{F} &= \frac{1}{2}\mathbf{k}\left[\partial_x\left(\frac{-x}{x^2+y^2}\right) + \partial_y\left(\frac{y}{x^2+y^2}\right)\right] \\
&= \frac{1}{2}\mathbf{k}\left[-\frac{1}{x^2+y^2} + \frac{2x^2}{(x^2+y^2)^2} - \frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2}\right] \\
&= 0.
\end{aligned} \tag{23}$$

To show that $\mathbf{F}$ is not conservative consider $\oint_C \mathbf{F}\cdot\mathbf{dl}$ where C is the unit circle. Using the obvious parametrization

$$\begin{aligned}
\oint_C \mathbf{F}\cdot\mathbf{dl} &= \int_0^{2\pi} du\ \left(-\sin^2 u - \cos^2 u\right) \\
&= -2\pi \neq 0,
\end{aligned} \tag{24}$$

therefore $\mathbf{F}$ is not conservative.

The domain $x^2+y^2 \geq \frac{1}{2}$, $y \geq 0$ is simply connected and $\mathbf{F}$ is irrotational and smooth is the domain. Thus $\mathbf{F}$ is conservative.

Write $\mathbf{F} = \nabla\phi$. Seek a $\phi(x, y)$ such that

$$\frac{\partial\phi}{\partial x} = \frac{y}{x^2+y^2}, \qquad \frac{\partial\phi}{\partial y} = -\frac{x}{x^2+y^2}. \tag{25}$$

Integrate first equation by treating y as a constant

$$\phi(x, y) = y\int\frac{dx}{x^2+y^2} = \tan^{-1}\frac{x}{y} + C(y). \tag{26}$$

Assume that x and y are non-negative, then

$$\tan^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x} = \frac{\pi}{2},$$

so that $\phi(x,y) = -\tan^{-1}\frac{y}{x}+$ a possibly y-dependent constant. However it is easy to check that $\phi = -\tan^{-1}\frac{y}{x}$ satisfies $\frac{\partial\phi}{\partial y} = -\frac{x}{x^2+y^2}$. Clearly, $\tan^{-1}\frac{y}{x}$ is the usual polar angle θ, that is $\phi = -\theta$.

Can try to extend this back to the original domain $x^2+y^2 \geq \frac{1}{2}$, but ϕ will suffer a branch cut discontinuity at, say $\theta = \frac{3}{2}\pi$.

6. Find the flux of $\mathbf{F} = e^{-y}\mathbf{i} - y\mathbf{j} + x\sin z\mathbf{k}$ across the portion of the paraboloid

$$\mathbf{r}(u,v) = 2\cos v\mathbf{i} + \sin v\mathbf{j} + u\mathbf{k} \tag{27}$$

with $0 \leq u \leq 5$ and $0 \leq v \leq 2\pi$, oriented to give a positive answer.

Solution:

NOTE: this is not really a paraboloid, there was a mix-up with a different question! Anyhow, using the given parameterization, we have

$$\begin{aligned}\frac{\partial\mathbf{r}}{\partial u} &= \mathbf{k}\\ \frac{\partial\mathbf{r}}{\partial v} &= -2\sin v\mathbf{i} + \cos v\mathbf{j}\end{aligned} \tag{28}$$

giving

$$\frac{\partial\mathbf{r}}{\partial u}\times\frac{\partial\mathbf{r}}{\partial v} = \begin{pmatrix} -\cos v\\ -2\sin v\\ 0\end{pmatrix} \tag{29}$$

On the surface

$$\mathbf{F}(\mathbf{r}(u,v)) = e^{-\sin v}\mathbf{i} - \sin v\mathbf{j} + 2\cos v\sin u\mathbf{k} \tag{30}$$

and the flux is

$$\phi = \int_0^5 du\int_0^{2\pi} dv\left(-\cos v e^{-\sin v} + 2\sin^2 v\right) \tag{31}$$

Now,

$$-\int_0^{2\pi} dv\cos v e^{-\sin v} = \int_0^{2\pi} dv\frac{d}{dv}e^{-\sin v} = e^{-\sin v}|_0^{2\pi} = 1-1=0 \tag{32}$$

This leaves the other bit of the integral, which we do using the usual

$$2\sin^2 x = 1-\cos 2x \tag{33}$$

giving

$$\phi = 10\pi \tag{34}$$

which is a positive answer, so the orientation was chosen correctly.

7. Use Green's Theorem to evaluate

$$\oint_c (y^2 dx + x^2 dy) \tag{35}$$

where C is the square with vertice $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$ and oriented anti-clockwise.

Solution:

By Green's theorem

$$\oint_c (y^2 dx + x^2 dy) = \int_0^1 dx \int_0^1 dy(2x - 2y) = \int_0^1 dx(2x - 1) = 0 \tag{36}$$

8. Calculate directly and using Stokes' Theorem

$$\int_S \mathbf{F} \cdot \mathbf{dS} \tag{37}$$

where $\mathbf{F} = (z - y)\mathbf{i} + (z + x)\mathbf{j} - (x + y)\mathbf{k}$ and S is the paraboloid $z = 9 - x^2 - y^2$ oriented upwards with $z > 0$.

Solution:

So, to calculate directly, choose some parameterization

$$\mathbf{r} = \rho \cos\phi \mathbf{i} + \rho \sin\phi \mathbf{j} + (9 - \rho^2)\mathbf{k} \tag{38}$$

works. Now

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial \phi} &= -\rho \sin\phi \mathbf{i} + \rho \cos\phi \mathbf{j} \\ \frac{\partial \mathbf{r}}{\partial \rho} &= \cos\phi \mathbf{i} + \sin\phi \mathbf{j} - 2\rho \mathbf{k} \end{aligned} \tag{39}$$

and, choosing the other order to make the normal point upwards,

$$\frac{\partial \mathbf{r}}{\partial \rho} \times \frac{\partial \mathbf{r}}{\partial \phi} = \begin{pmatrix} 2\rho^2 \cos\phi \\ 2\rho^2 \sin\phi \\ \rho \end{pmatrix} \tag{40}$$

Now, writing this as $(2\rho x, 2\rho y, \rho)$ and doing the dot product with $\mathbf{F}$ we are left with only terms which are linear in x or y and since the ϕ integral goes all the way around, we see the answer is zero.

Next, using Stokes

$$\int_S \operatorname{curl}\mathbf{A}\cdot\mathbf{dS} = \oint_c \mathbf{A}\cdot\mathbf{dl} \tag{41}$$

hence, to apply Stokes, we have to write $\mathbf{F}$ as $\operatorname{curl}\mathbf{A}$, in other words, find a vector potential for $\mathbf{F}$. It is easy to check that $\operatorname{div}\mathbf{F} = 0$ so this should be possible. We will use the formula that was used to prove the existence of vector potential for divergenceless fields on star-shapped domains. Hence

$$\begin{aligned}
\mathbf{A} &= \int_0^1 \mathbf{F}(t\mathbf{r})\times t\mathbf{r} \\
&= \frac{1}{3}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ z-y & z+x & -(x+y) \\ x & y & z \end{vmatrix} \\
&= \frac{1}{3}\begin{pmatrix} z^2+xz+xy+y^2 \\ -x^2-xy-z^2+yz \\ zy-y^2-xz-x^2 \end{pmatrix}
\end{aligned} \tag{42}$$

where I got the third by noting that the overall factor of t^2 came out of the determinant, and then integrating it. Since this formula is complicated it would certainly be a good idea to check $\mathbf{F} = \nabla\times\mathbf{F}$.

Now, to apply Stoke's theorem:

$$\int_S \operatorname{curl}\mathbf{A}\cdot\mathbf{dS} = \oint_c \mathbf{A}\cdot\mathbf{dl} \tag{43}$$

where C is the circle of radius three around the origin in the xy-plane: $x^2 + y^2 = 9$ and $z = 0$. We parameterize with

$$\mathbf{r} = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} \tag{44}$$

so that

$$\frac{d\mathbf{r}}{dt} = -3\sin t\mathbf{i} + 3\cos t\mathbf{j} \tag{45}$$

Restricting $\mathbf{A}$ to the curve and doing the dot product gives

$$\oint_c \mathbf{A} \cdot \mathbf{dl} = 9 \int_0^{2\pi} (-cs^2 - s^3 - c^3 - c^2 s) dt = 0 \tag{46}$$

where $c = \cos t$ and $s = \sin t$ and we are using the usual anti-symmetry argument that odd powers of sine and cosine integrate to zero over their entire period.