MA2331 Tutorial Sheet 5.[1]

4 December 2014
(Due 12 December 2014 in class)

Useful facts:

- To evaluate the line integral for a parameterized curve $C : t \to \mathbf{r}(t)$:

$$\int_c \mathbf{F} \cdot \mathbf{dl} = \int_{t_1}^{t_2} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt \tag{1}$$

  where t_1 and t_2 are the parameter values corresponding to the beginning and end of the curve.

- To evaluate the surface integral for a parameterized surface $S : (u, v) \to \mathbf{r}(u, v)$:

$$\int \int_S \mathbf{F} \cdot \mathbf{dA} = \int \int_S \mathbf{F} \cdot \left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right) dudv \tag{2}$$

- Stokes' theorem: for an orientable piecewise smooth surface S with an orientable piecewise smooth boundy C oriented so that $\mathbf{n} \times \mathbf{dl}$ points into S with $\mathbf{n}$ the normal and $\mathbf{dl}$ a tangent to C and $\mathbf{F}$ a vector field defined in a region containing S, then

$$\int_S \text{curl}\, \mathbf{F} \cdot \mathbf{dA} = \int_C \mathbf{F} \cdot \mathbf{dl} \tag{3}$$

- Green's theorem on the plane: led D be a region in the xy-plane bounded by a piecewise continuous curve C, if $f(x, y)$ and $g(x, y)$ have continuous first derivatives

$$\int_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dxdy = \int_C (f dx + g dy) \tag{4}$$

- A vector field $\mathbf{F}$ is **conservative** if $\mathbf{F}$ =grad ϕ for some scalar field ϕ. ϕ is often called a **potential** for $\mathbf{F}$.

- A vector field is path-independent if its line integral between any two points is independent of the path.

- An irrotational field on a simply connected domain is conservative.

- To find a potential by integration: let $\partial_x \phi = F_1$ and integrate to find ϕ determined up to an arbitrary function of y and z, substitute back into $\partial_y \phi = F_2$ to determine it up to an arbitary function of z and then determine this, up to an arbitrary constant, by substituting into $\partial_z \phi = F_3$

[1]Stefan Sint, `sint@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~sint/MA2331/MA2331.html`

Questions

1. Compute the line integrals:

   (a) $\int_C \; (dx \; xy \; + \frac{1}{2}dy \; x^2 \; + dz)$ where C is the line segment joining the origin and the point $(1, 1, 2)$.

   (b) $\int_C \; (dx \; yz \; + dy \; xz \; + dz \; yx^2)$ where C is the same line as in the previous part

   (**2 marks**)

2. For each of the following vector fields compute the line integral $\oint_C \mathbf{F} \cdot \mathbf{dl}$ where C is the unit circle in the xy-plane taken anti-clockwise.

   (a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$

   (b) $\mathbf{F} = y\mathbf{i} - x^2 y\mathbf{j}$.

   (**2 marks**)

3. Evaluate the line integrals $\int_C \mathbf{F} \cdot \mathbf{dl}$ for

   (a) $\mathbf{F} = (x^2 y, 4, 0)$ with C given by $\mathbf{r}(t) = (\exp(t), \exp(-t), 0)$ with t going from zero to one;

   (b) $\mathbf{F} = (z, x, y)$ with C given by $\mathbf{r}(t) = (\sin t, 3\sin t, \sin^2 t)$ with t going from zero to $\pi/2$.

   (**2 marks**)

4. For each of these fields determine if $\mathbf{F}$ is conservative, if it is, by integration or otherwise, find a potential: ϕ such that $\mathbf{F} = \nabla\phi$.

   (a) $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$

   (b) $\mathbf{F} = 3y^2\mathbf{i} + 6xy\mathbf{j}$

   (c) $\mathbf{F} = e^x \cos y\mathbf{i} - e^x \sin y\mathbf{j}$

   (d) $\mathbf{F} = (\cos y + y\cos x)\mathbf{i} + (\sin x - x\sin y)\mathbf{j}$

   (**2 marks**)

5. Consider the 'point vortex' vector field

   $$\mathbf{F} = \frac{y}{x^2 + y^2}\mathbf{i} - \frac{x}{x^2 + y^2}\mathbf{j}.$$

   Show that curl $\mathbf{F} = 0$ away from the z-axis. Establish that $\mathbf{F}$ is *not* conservative in the (non simply-connected) domain $x^2 + y^2 \geq \frac{1}{2}$. Is $\mathbf{F}$ conservative in the domain defined by $x^2 + y^2 \geq \frac{1}{2}$, $y \geq 0$? If so obtain a scalar potential for $\mathbf{F}$.

   (**2 marks**)

6. Find the flux of $\mathbf{F} = e^{-y}\mathbf{i} - y\mathbf{j} + x\sin z\mathbf{k}$ across the portion of the paraboloid

$$\mathbf{r}(u,v) = 2\cos v\mathbf{i} + \sin v\mathbf{j} + u\mathbf{k} \tag{5}$$

with $0 \leq u \leq 5$ and $0 \leq v \leq 2\pi$, oriented to give a positive answer.

(**2 marks**)

7. Use Green's Theorem to evaluate

$$\oint_c (y^2dx + x^2dy) \tag{6}$$

where C is the square with vertice $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$ and oriented anti-clockwise.

(**2 marks**)

8. Calculate directly and using Stokes' Theorem

$$\int_S \mathbf{F} \cdot \mathbf{dS} \tag{7}$$

where $\mathbf{F} = (z-y)\mathbf{i} + (z+x)\mathbf{j} - (x+y)\mathbf{k}$ and S is the paraboloid $z = 9 - x^2 - y^2$ oriented upwards with $z > 0$.

(**2 marks**)