MA2331 Tutorial Sheet 4, Solutions.[1]

17 November 2014
(Due 25 November 2014 in class)

Questions

1. Check that the Jacobian for the transformation from cartesian to spherical coordinates is
$$J = r^2 \sin\theta. \tag{1}$$
Consider the hemisphere defined by
$$\sqrt{x^2+y^2+z^2} \le 1, \qquad z \ge 0 \tag{2}$$
Using spherical coordinates compute its volume and centroid[2].

Solution:

The change of variables is
$$x = r\cos\phi\sin\theta, \quad y = r\sin\phi\sin\theta, \quad z = r\cos\theta \tag{3}$$
and the functional determinant is then
$$\begin{aligned} J &= \begin{vmatrix} \partial_r x & \partial_\phi x & \partial_\theta x \\ \partial_r y & \partial_\phi y & \partial_\theta y \\ \partial_r z & \partial_\phi z & \partial_\theta z \end{vmatrix} = \begin{vmatrix} \cos\phi\sin\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta \\ \cos\theta & 0 & -r\sin\theta \end{vmatrix} \\ &= -r^2\cos^2\phi\sin^3\theta - r^2\sin^2\phi\sin\theta - r^2\cos^2\phi\cos^2\theta\sin\theta \\ &= -r^2\sin\theta \end{aligned} \tag{4}$$
Hence we confirm the result for J up to a sign, $J = -r^2\sin\theta$. In fact there is a sign ambiguity as long as one does not fix the order of the coordinates e.g. (r,ϕ,θ) vs. (r,θ,ϕ), as this determines the column order in the determinant. For the change of variables in the integral this does not matter as this involves the absolute value $|J|$.

[1] Stefan Sint, `sint@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~sint/MA2331/MA2331.html`

[2] Recall that the centroid is the geometric mean, which coincides with the centre of mass provided the mass density is constant.

To compute the centroid of the half sphere, the symmetry implies that the centroid $\mathbf{r}_0$ defined by

$$\mathbf{r}_0 = \frac{\int_{B_h}(x,y,z)d\mu_{xyz}}{\int_{B_h} d\mu_{xyz}}, \tag{5}$$

where B_h is the half sphere. Changing to spherical coordinates, the denominator integral is given by

$$\int_{B_h} d\mu_{xyz} = \int_0^1 r^2 dr \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin\theta d\theta = 2\pi \int_0^1 r^2 dr = \frac{2}{3}\pi \tag{6}$$

as expected for the half sphere. Computing the numerator integrals we get 0 for x_0 and y_0 as this is proportional to the integrals of either $\cos\phi$ or $\sin\phi$ over the full period from 0 to 2π. For z_0 one then obtains

$$\begin{aligned} z_0 &= \frac{3}{2\pi}\int_0^1 r^3 dr \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin\theta\cos\theta d\theta \\ &= \frac{3}{2\pi}2\pi\frac{1}{4}\int_0^{\pi/2} \sin\theta\cos\theta d\theta \\ &= -\frac{3}{4}\int_1^0 u du = \frac{3}{8} \end{aligned} \tag{7}$$

Hence the centroid has coordinates $(0,0,3/8)$

2. Show $\operatorname{div}\mathbf{r} = 3$ and $\operatorname{grad}|\mathbf{r}| = \mathbf{r}/|\mathbf{r}|$.

   Solution:

   We have

$$\nabla\mathbf{r} = \partial_x x + \partial_y y + \partial_z z = 1+1+1 = 3 \tag{8}$$

   For the gradient we need:

$$\partial_x r = \frac{1}{2\sqrt{x^2+y^2+z^2}}2x = \frac{x}{r} \tag{9}$$

$$\partial_y r = \frac{1}{2\sqrt{x^2+y^2+z^2}}2y = \frac{y}{r} \tag{10}$$

$$\partial_z r = \frac{1}{2\sqrt{x^2+y^2+z^2}}2z = \frac{z}{r} \tag{11}$$

so that

$$\nabla r = \frac{(x,y,z)}{r} = \frac{\mathbf{r}}{r}. \tag{12}$$

3. Find $\nabla(1/|\mathbf{r}|)$. *Solution:*

   We have

$$\nabla(1/r) = (\partial_x(1/r), \partial_y(1/r), \partial_z(1/r) = -\frac{1}{r^2}\nabla r \tag{13}$$

   Using the result from the preceding question this gives $-\mathbf{r}/r^3$.

4. Show grad $f(r) = f'(r)\hat{\mathbf{r}}$. If $\mathbf{F}(r) = f(r)\mathbf{r}$ find div $\mathbf{F}(r)$. Find div grad $f(r)$.

   Solution:

   By the chain rule and question 2 we have

$$\nabla f(r) = (\partial_x f(r), \partial_y f(r), \partial_z f(r)) = f'(r)\nabla r = f'(r)\hat{\mathbf{r}} \tag{14}$$

   Next, we have, again using question 2,

$$\begin{aligned}\nabla \cdot f(r)\mathbf{r} &= \partial_x(xf(r)) + \partial_y(yf(r)) + \partial_z(zf(r) \\ &= 3f(r) + f'(r)\mathbf{r}\cdot\nabla r \\ &= 3f(r) + rf'(r)\end{aligned} \tag{15}$$

   Finally, using these results we have

$$\begin{aligned}\nabla \cdot \nabla f(r) &= \nabla \cdot f'(r)\frac{\mathbf{r}}{r} \\ &= \partial_x(xf'(r)/r) + \partial_y(yf'(r)/r) + \partial_z(zf'(r)/r)\end{aligned} \tag{16}$$

   By the chain rule one then finds

$$\nabla \cdot \nabla f(r) = 3f'(r)/r + ((f'(r)/r)'\frac{x^2+y^2+z^2}{r} = f''(r) + 2\frac{f'(r)}{r}. \tag{17}$$

5. Show that away from the origin the vector field

$$\mathbf{F} = \frac{\hat{\mathbf{r}}}{r^2} = \frac{\mathbf{r}}{r^3} \tag{18}$$

   is irrotational.

Solution:

We need to calculate the curl of this vector field and show that it vanishes away from the origin.

$$\begin{aligned} \nabla \times \frac{\mathbf{r}}{r^2} &= \mathbf{i}(z\partial_y r^{-2} - y\partial_z r^{-2}) \\ &\quad +\mathbf{j}(x\partial_z r^{-2} - z\partial_x r^{-2}) \\ &\quad +\mathbf{k}(y\partial_x r^{-2} - x\partial_y r^{-2}) \end{aligned} \tag{19}$$

Now, since

$$\partial_x r^{-2} = -2r^{-3}\partial_x r = -2\frac{x}{r^4}, \quad \partial_y r^{-2} = -2\frac{y}{r^4}, \quad \partial_z r^{-2} = -2\frac{z}{r^4} \tag{20}$$

we have for the first component,

$$\left(\nabla \times \frac{\mathbf{r}}{r^2}\right)_1 = z\partial_y r^{-2} - y\partial_z r^{-2} = -2r^{-4}(zy - yz) = 0, \tag{21}$$

and similarly for the other terms, so that the curl indeed vanishes for $r \neq 0$.

6. Prove the identities

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{F}) &= 0 \qquad (22) \\ \nabla \times (\nabla \times \mathbf{F}) &= \nabla(\nabla \cdot \mathbf{F}) - \triangle \mathbf{F}. \qquad (23) \end{aligned}$$

Solution:

By the definition of the divergence we have

$$\nabla \cdot (\nabla \times \mathbf{F}) = \partial_x(\nabla \times \mathbf{F})_1 + \partial_y(\nabla \times \mathbf{F})_2 + \partial_z(\nabla \times \mathbf{F})_3. \tag{24}$$

Inserting the definition of the curl

$$\nabla \cdot (\nabla \times \mathbf{F}) = \partial_x(\partial_y F_3 - \partial_z F_2) + \partial_y(\partial_z F_1 - \partial_x F_3) + \partial_z(\partial_x F_2 - \partial_y F_1) \tag{25}$$

For twice continuously differentiable vector fields the order of the derivatives does not matter and, collecting the terms acting on $F_{1,2,3}$, one finds e.g. for F_3,

$$(\partial_x\partial_y - \partial_y\partial_x)F_3 = 0 \tag{26}$$

and the same for the other components.

To check the other identity we compute the first component on either side, the other components work analogously. The first component on the lhs is

$$(\nabla \times (\nabla \times \mathbf{F}))_1 = \partial_y(\nabla \times \mathbf{F})_3 - \partial_z(\nabla \times \mathbf{F})_2 \tag{27}$$

Now we insert the components of the second curl, so that

$$(\nabla \times (\nabla \times \mathbf{F}))_1 = \partial_y(\partial_x F_2 - \partial_y F_1) - \partial_z(\partial_z F_1 - \partial_x F_3) \tag{28}$$

This has to be compared to the first component on the rhs, given by

$$\partial_x(\partial_x F_1 + \partial_y F_2 + \partial_z F_3) - (\partial_x^2 + \partial_y^2 + \partial_z^2)F_1 = \partial_x(\partial_y F_2 + \partial_z F_3) - (\partial_y^2 + \partial_z^2)F_1, \tag{29}$$

which is indeed the same given that the order of derivatives does not matter (assuming that we have a twice continuously differentiable vector field). The other components can be checked analogously, thus establishing the vector identity.