MA2331 Tutorial Sheet 3, Solutions[1]

31 October 2014
(Due 10 November 2014 in class)

Questions

1. Compute

   (a) $\int_{-\infty}^{\infty} dx\ x^2\ \delta(x-3)$;

   (b) $\int_{-\infty}^{\infty} dx\ \delta(x^2+x)$;

   (c) $\int_{0}^{\infty} dx\ e^{-ax}\delta(\cos x)$, where a is a constant;

   (d) $\int_{0}^{\infty} dx\ \delta(e^{ax}\cos x)$, where a is a constant.

   (**4 marks**)

   Solution:

   (a)
$$\int_{-\infty}^{\infty} dx\ x^2\ \delta(x-3) = 3^2 = 9 \tag{1}$$

   (b) Here $h(x) = x^2 + x = x(x+1)$ so zeros for $x = 0$ and $x = -1$. $h'(x) = 2x+1$, evaluated at the zeros gives $h'(0) = 1$ and $h'(-1) = -1$, in both cases $|h'(0)| = |h'(-1)| = 1$ and

$$\int_{-\infty}^{\infty} dx\ \delta(x^2+x) = \int_{-\infty}^{\infty} dx\ \delta(x) + \int_{-\infty}^{\infty} dx\ \delta(x+1) = 1+1 = 2 \tag{2}$$

   (c) The positive zeros of $\cos x$ are at $x_k = (k+1/2)\pi$ for $k = 0, 1, 2, \ldots$ and we get

$$\int_{0}^{\infty} dx\ e^{-ax}\delta(\cos x) = \int_{0}^{\infty} dx\ e^{-ax} \sum_{k=0}^{\infty} \frac{\delta(x-x_k)}{|\sin(x_k)|} \tag{3}$$

[1]Stefan Sint, `sint@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~sint/MA2331/MA2331.html`

Now, $|\sin(x_k)| = 1$ for all k so that we get

$$\int_0^\infty dx\, e^{-ax}\delta(\cos x) = \sum_{k=0}^\infty e^{-ax_k} = e^{-a\pi/2}\sum_{k=0}^\infty \left(e^{-a\pi}\right)^k = \frac{e^{-a\pi/2}}{1-e^{-a\pi}} = \frac{1}{2\sinh(a\pi/2)} \tag{4}$$

where we have used the geometric series and assumed $a > 0$ as otherwise the series would diverge.

(d) We have $h(x) = e^{ax}\cos x$ so the zeros are again $x_k = (k+1/2)\pi$ for $k = 0, 1, 2, \ldots$ and $h'(x) = ah(x) - e^{ax}\sin x$, so that $|h'(x_k)| = e^{ax_k}$, and we obtain

$$\int_0^\infty dx\, \delta(e^{ax}\cos x) = \sum_{k=0}^\infty \frac{1}{e^{ax_k}} = e^{-a\pi/2}\sum_{k=0}^\infty \left(e^{-a\pi}\right)^k \tag{5}$$

which is the same series as above, again $a > 0$ has to be assumed and one gets the very same result as in the previous case.

2. Compute the double integral for finite constants $a, b \geq 1$,

$$f(a,b) := \int_1^a dx \int_1^b dy \frac{x-y}{(x+y)^3}$$

How does $f(a,b)$ relate to $f(b,a)$? Investigate the double limits

$$\lim_{a\to\infty}\lim_{b\to\infty} f(a,b), \qquad \lim_{b\to\infty}\lim_{a\to\infty} f(a,b)$$

Does the order of the limits matter?

(**2 marks**)

Solution: Straightforward integration gives:

$$f(a,b) = \frac{-b}{a+b} + \frac{b}{1+b} + \frac{1}{a+1} - \frac{1}{2}$$

Exchanging a with b shows that

$$f(a,b) = -f(b,a)$$

The limits do not commute, in fact due to the above anti-symmetry one has

$$\lim_{a\to\infty}\lim_{b\to\infty} f(a,b) = -\lim_{a\to\infty}\lim_{b\to\infty} f(b,a) = -\lim_{b\to\infty}\lim_{a\to\infty} f(a,b),$$

which means equality is only possible if the limits vanish, which is not the case as

$$\lim_{a\to\infty}\lim_{b\to\infty} f(a,b) = -\frac{1}{2}$$

3. Calculate the volume of a ball with radius R in 3 dimensions,

$$B = \{(x,y,z)|x^2+y^2+z^2 \le R^2\}.$$

Use Fubini's theorem to successively integrate over the Cartesian coordinates x,y,z (i.e. do *not* change variables!).

(**2 marks**)

Solution:

We do the reduction in 2 steps: first we fix (x,y) and integrate z over $B_{x,y} = [-\sqrt{R^2-x^2-y^2}, \sqrt{R^2-x^2-y^2}]$ with $B' = \{(x,y)|x^2+y^2 \le R^2\}$. In the second step we fix x and integrate y over $B'_x = [-\sqrt{R^2-x^2}, \sqrt{R^2-x^2}]$ and finally x over $B'' = [-R,R]$. Together we thus have

$$\int_B d\mu = \int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} dz \tag{6}$$

Carrying out the z-integration, we first get

$$\int_B d\mu = \int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 2\sqrt{R^2-x^2-y^2}dy \tag{7}$$

To carry out the y integral it is useful to change variables $w = y/\sqrt{R^2-x^2}$, so that

$$\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 2\sqrt{R^2-x^2-y^2}dy = 2(R^2-x^2)\int_{-1}^{1} dw\sqrt{1-w^2} = 2(R^2-x^2)\frac{\pi}{2} = \pi(R^2-x^2), \tag{8}$$

where the w-integral is easily solved using the trigonometric change of variables $w = \sin u$. Now it remains to integrate this over x:

$$\int_B d\mu = \int_{-R}^{R} \pi(R^2-x^2) = \pi R^3(2-2/3) = \frac{4}{3}\pi R^3, \tag{9}$$

hardly a surprise!

4. Calculate the following integrals over the given sets using the reduction by Fubini (it helps to make a drawing):

   (a)
$$\int_A d\mu_{xy}, \qquad \int_A x^2 d\mu_{xy}, \qquad A = \{(x,y)||x| + |y| \leq 1\}$$

   (**2 marks**)

   Solution:

   The set A is the square with the corner points $(1,0),(0,1),(-1,0),(0,-1)$, the area should therefore come out as $(\sqrt{2})^2 = 2$. We first fix x and integrate y over $A_x = \{y||y| \leq 1 - |x|\} = [-1+|x|, 1-|x|]$. and x is to be integrated over $A' = [-1,1]$. Hence we have

$$\int_A d\mu_{xy} = \int_{A'} (\int_{A_x} dy)dx = \int_{-1}^{1} dx \int_{-1+|x|}^{1-|x|} dy = \int_{-1}^{1} 2(1-|x|) = 4\int_0^1 (1-x) = 4\frac{1}{2} = 2 \tag{10}$$

   where we have used that $|x|$ is an even function. For the second integral we obtain

$$\int_A x^2 d\mu_{xy} = \int_{-1}^{1} dx x^2 \int_{-1+|x|}^{1-|x|} = \int_{-1}^{1} 2x^2(1-|x|) = 4\int_0^1 (x^2 - x^3) = \frac{4}{3} - 1 = \frac{1}{3}. \tag{11}$$

   (b)
$$\int_A d\mu_{xy}, \qquad \int_A xy d\mu_{xy}, \qquad A = \{(x,y)|0 \leq x \leq y \leq 1\}$$

   (**2 marks**)

   Solution:

   The set A is the triangle with corners at $(0,0),(1,1)$ and $(0,1)$. We first fix x and integrate y over the set $A_x = [x,1]$ and then x over $A' = [0,1]$,

$$\int_A d\mu_{xy} = \int_0^1 dx \int_x^1 dy = \int_0^1 dx(1-x) = 1 - \frac{1}{2} = \frac{1}{2} \tag{12}$$

$$\int_A xy d\mu_{xy} = \int_0^1 dx x \int_x^1 y dy = \int_0^1 dx x(\frac{1}{2} - \frac{1}{2}x^2) = \frac{1}{4} - \frac{1}{8} = \frac{1}{8} \tag{13}$$

(c)

$$\int_A d\mu_{xy}, \qquad \int_A \exp(y^2) d\mu_{xy}, \qquad A = \{(x,y)||x| \le y \le 1\}$$

(**2 marks**)
*Solution:*The set A corresponds to a triangle with corners at $(0,0)$, $(1,1)$ and $(-1,1)$. While the order of integrations is irrelevant in principle, in practice the calculation of the second integral is much easier if x is integrated first over $A_y = [-y, y]$, then y over $A' = [0,1]$. Hence

$$\int_A d\mu_{xy} = \int_0^1 dy \int_{-y}^{y} dx = \int_0^1 2y = 2\frac{1}{2} = 1, \tag{14}$$

and

$$\int_A \exp(y^2) d\mu_{xy} = \int_0^1 dy \exp(y^2) \int_{-y}^{y} dx = \int_0^1 2y \exp(y^2) dy = \exp(y^2)|_0^1 = e-1 \tag{15}$$

(d)

$$\int_A d\mu_{xy}, \qquad \int_A x d\mu_{xy}, \qquad A = \{(x,y)|x^2 + y^2 \le R^2 \text{ and } x \ge 0\}$$

(**2 marks**)
Solution:(Remark: I meant to write R^2 in the question; to remain consistent with that typo I set $r = \sqrt{R}$ in the following.) The set A is the right half disk with radius $r = \sqrt{R}$. We first fix y and integrate x over $A_y = [0, \sqrt{r^2 - y^2}]$ then y over $A' = [-r, r]$. We thus have

$$\int_A d\mu_{xy} = \int_{-r}^{r} dy \int_0^{\sqrt{r^2-y^2}} dx = \int_{-r}^{r} dy \sqrt{r^2 - y^2} = \frac{\pi}{2} r^2, \tag{16}$$

which is best solved by the trigonometric substitution $y = r \sin u$. The other integral is

$$\int_A x d\mu_{xy} = \int_{-r}^{r} dy \int_0^{\sqrt{r^2-y^2}} x dx = 2 \int_0^r dy \frac{1}{2}(r^2 - y^2) = r^3 - \frac{1}{3} r^3 = \frac{2}{3} r^3, \tag{17}$$

where we have used that $r^2 - y^2$ is an even function.