MA2331 Tutorial Sheet 3.[1]

31 October 2014
(Due 10 November 2014 in class)

Useful facts:

- The Dirac delta function:

$$\int_{-\infty}^{\infty} dx\, f(x)\delta(x-a) = f(a)$$

- If $h(x)$ is a continuous function, differentiable with non-vanishing derivatives at its zeros, x_i, then

$$\int_{-\infty}^{\infty} dx\, f(x)\delta[h(x)] = \sum_{x_i:h(x_i)=0} \frac{f(x_i)}{|h'(x_i)|}$$

Questions

1. Compute
   - $\int_{-\infty}^{\infty}\ dx\ x^2\ \delta(x-3)$;
   - $\int_{-\infty}^{\infty}\ dx\ \delta(x^2+x)$;
   - $\int_{0}^{\infty}\ dx\ e^{-ax}\delta(\cos x)$, where a is a constant;
   - $\int_{0}^{\infty}\ dx\ \delta(e^{ax}\cos x)$, where a is a constant.

   (**4 marks**)

2. Compute the double integral for finite constants $a, b \geq 1$,

$$f(a,b) := \int_1^a dx \int_1^b dy \frac{x-y}{(x+y)^3}$$

   How does $f(a,b)$ relate to $f(b,a)$? Investigate the double limits

$$\lim_{a\to\infty}\lim_{b\to\infty} f(a,b), \qquad \lim_{b\to\infty}\lim_{a\to\infty} f(a,b)$$

   Does the order of the limits matter?

   (**2 marks**)

[1] Stefan Sint, `sint@maths.tcd.ie`, see also `http://www.maths.tcd.ie/~sint/MA2331/MA2331.html`

3. Calculate the volume of a ball with radius R in 3 dimensions,

$$B = \{(x, y, z) | x^2 + y^2 + z^2 \leq R^2\}.$$

Use Fubini's theorem to successively integrate over the Cartesian coordinates x,y,z (i.e. do *not* change variables!).

(**2 marks**)

4. Calculate the following integrals over the given sets using the reduction by Fubini (it helps to make a drawing):

(a)

$$\int_A d\mu_{xy}, \qquad \int_A x^2 d\mu_{xy}, \qquad A = \{(x, y) || x| + |y| \leq 1\}$$

(**2 marks**)

(b)

$$\int_A d\mu_{xy}, \qquad \int_A xy d\mu_{xy}, \qquad A = \{(x, y) | 0 \leq x \leq y \leq 1\}$$

(**2 marks**)

(c)

$$\int_A d\mu_{xy}, \qquad \int_A \exp(y^2) d\mu_{xy}, \qquad A = \{(x, y) || x| \leq y \leq 1\}$$

(**2 marks**)

(d)

$$\int_A d\mu_{xy}, \qquad \int_A x d\mu_{xy}, \qquad A = \{(x, y) | x^2 + y^2 \leq R \text{ and } x \geq 0\}$$

(**2 marks**)