Linear Algebra

Course 211

David Simms

School of Mathematics

Trinity College

Dublin

Original typesetting

Brendan Griffin.

Alterations

Geoff Bradley.

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Chapter 1

Vector Spaces

Recall that in course 131 you studied the notion of a *linear vector space*. In that course the scalars were *real numbers*. We will study the more general case, where the set of scalars is any field K. For example $\mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}/(p)$.

Definition. Let K be a field. A set M is called a *vector space over the field* K (or a K*-vector space*) if

(i) an operation

$$\begin{aligned} M \times M &\to M \\ (x, y) &\mapsto x + y \end{aligned}$$

is given, called *addition of vectors*, which makes M into a commutative group;

(ii) an operation

$$\begin{aligned} K \times M &\to M \\ (\lambda, x) &\mapsto \lambda x \end{aligned}$$

is given, called *multiplication of a vector by a scalar*, which satisfies:

(a) $\lambda(x + y) = \lambda x + \lambda y$,

(b) $(\lambda + \mu)x = \lambda x + \mu x$,

(c) $\lambda(\mu x) = (\lambda\mu)x$,

(d) $1x = x$

for all $\lambda, \mu \in K,\ x, y \in M,$ where 1 is the unit element of the field K.

The elements of M are then called the *vectors*, and the elements of K are called the *scalars* of the given K-vector space M.

Examples:

1. The set of 3-dimensional geometrical vectors (as in 131) is a real vector space ($\mathbb{R}$-vector space).

2. The set $\mathbb{R}^n$ (as in 131) is a real vector space.

3. If K is any field then the following are K-vector spaces:

   (a) $K^n = \{(\alpha_1, \dots, \alpha_n) : \alpha_1, \dots, \alpha_n \in K\}$, with vector addition:

   $$(\alpha_1, \dots, \alpha_n) + (\beta_1, \dots, \beta_n) = (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n),$$

   and scalar multiplication:

   $$\lambda(\alpha_1, \dots, \alpha_n) = (\lambda\alpha_1, \dots, \lambda\alpha_n).$$

   (b) The set $K^{m\times n}$ of $m \times n$ matrices (m rows and n columns) with entries in K (m, n fixed integers ≥ 1), with vector addition:

   $$\begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix} + \begin{pmatrix} \beta_{11} & \cdots & \beta_{1n} \\ \vdots & & \vdots \\ \beta_{m1} & \cdots & \beta_{mn} \end{pmatrix}$$
   $$= \begin{pmatrix} \alpha_{11} + \beta_{11} & \cdots & \alpha_{1n} + \beta_{1n} \\ \vdots & & \vdots \\ \alpha_{m1} + \beta_{m1} & \cdots & \alpha_{mn} + \beta_{mn} \end{pmatrix},$$

   and scalar multiplication:

   $$\lambda \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{pmatrix} = \begin{pmatrix} \lambda\alpha_{11} & \cdots & \lambda\alpha_{1n} \\ \vdots & & \vdots \\ \lambda\alpha_{m1} & \cdots & \lambda\alpha_{mn} \end{pmatrix}.$$

   (c) The set K^X of all maps from X to K (X a fixed non-empty set), with vector addition:

   $$(f + g)(x) = f(x) + g(x),$$

   and scalar multiplication:

   $$(\lambda f)(x) = \lambda(f(x))$$

   for all $x \in X,\ f, g \in K^X,\ \lambda \in K$.

Definition. Let $N \subset M$, and let M be a K-vector space. Then N is called a *K-vector subspace* of M if N is non-empty, and

(i) $x, y \in N \Rightarrow x + y \in N$ closed under addition;

(ii) $\lambda \in K,\ x \in N \Rightarrow \lambda x \in N$ closed under scalar multiplication.

Thus N is itself a K-vector space.

Examples:

1. $\{(\alpha, \beta, \gamma) : 3\alpha + \beta - 2\gamma = 0;\ \alpha, \beta, \gamma \in \mathbb{R}\}$ is a vector subspace of $\mathbb{R}^3$.

2. $\{\underline{v} : \underline{v}.\underline{n} = 0\}$, $\underline{n}$ fixed, is a vector subspace of the space of 3-dimensional geometric vectors (see Figure 1.1).

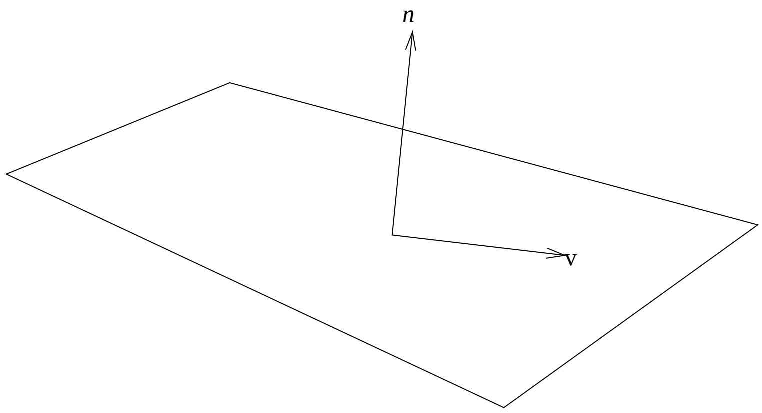


Figure 1.1

3. The set $C^0(\mathbb{R})$ of continuous functions is a real vector subspace of the set $\mathbb{R}^{\mathbb{R}}$ of all maps $\mathbb{R} \to \mathbb{R}$.

4. Let V be an open subset of $\mathbb{R}$. We denote by

   $C^0(V)$ the space of all continuous real valued functions on V,

   $C^r(V)$ the space of all real valued functions on V having continuous rth derivative,

   $C^\infty(V)$ the space of all real valued functions on V having derivatives of all r.

   Then

   $$C^\infty(V) \subset \cdots \subset C^{r+1}(V) \subset C^r(V) \subset \cdots \subset C^0(V) \subset \mathbb{R}^V$$

   is a sequence of real vector subspaces.

5. The space of solutions of the differential equation
$$\frac{d^2u}{dx^2} + w^2u = 0$$
is a real vector subspace of $C^\infty(\mathbb{R})$.

Definition. Let $u_1, \ldots, u_r$ be vectors in a K-vector space M, and let $\alpha_1, \ldots, \alpha_r$ be scalars. Then the vector
$$\alpha_1 u_1 + \cdots + \alpha_r u_r$$
is called a *linear combination* of $u_1, \ldots, u_r$. We write
$$\mathcal{S}(u_1, \ldots, u_r) = \{\alpha_1 u_1 + \cdots + \alpha_r u_r : \alpha_1, \ldots, \alpha_r \in K\}$$
to denote the set of all linear combinations of $u_1, \ldots, u_r$. $\mathcal{S}(u_1, \ldots, u_r)$ is a K-vector subspace of M, and is called the *subspace generated by* $u_1, \ldots, u_r$.

If $\mathcal{S}(u_1, \ldots, u_r) = M$, we say that $u_1, \ldots, u_r$ *generate* M (i.e. for each $x \in M$ there exists $\alpha_1, \ldots, \alpha_r \in K$ such that $x = \alpha_1 u_1 + \cdots + \alpha_r u_r$).

Examples:

1. The vectors $(1, 2), (-1, 1)$ generate $\mathbb{R}^2$ (see Figure 1.2), since
$$(\alpha, \beta) = \frac{\alpha + \beta}{3}(1, 2) + \frac{\beta - 2\alpha}{3}(-1, 1).$$
2. The functions $\cos \omega x$, $\sin \omega x$ generate the space of solutions of the differential equation:
$$\frac{d^2u}{dx^2} + w^2u = 0.$$

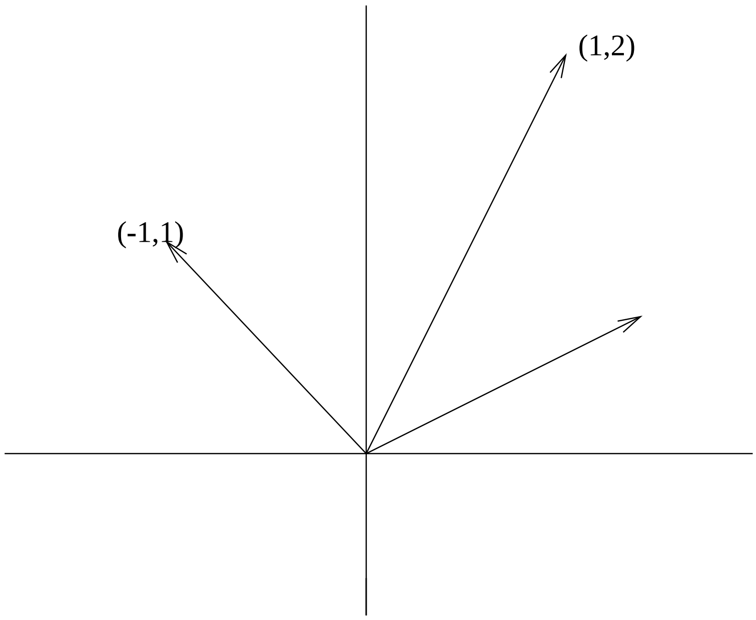


Figure 1.2

Definition. Let $u_1, \ldots, u_r$ be vectors in a K-vector space M. Then

(i) $u_1, \ldots, u_r$ are *linearly dependent* if there exist $\alpha_1, \ldots, \alpha_r \in K$ *not all zero* such that
$$\alpha_1 u_1 + \cdots + \alpha_r u_r = 0;$$

(ii) $u_1, \ldots, u_r$ are *linearly independent* if
$$\alpha_1 u_1 + \cdots + \alpha_r u_r = 0$$
implies that $\alpha_1, \ldots, \alpha_r$ *are all zero.*

Example: $\cos \omega x$, $\sin \omega x$ $(\omega \neq 0)$ are linearly independent functions in $C^\infty(\mathbb{R})$.

Proof of This $\triangleright$ Let
$$\alpha \cos \omega x + \beta \sin \omega x = 0; \quad \alpha, \beta \in \mathbb{R}$$
be the zero function. Put $x = 0 : \alpha = 0$; put $x = \frac{\pi}{2\omega} : \beta = 0$. $\triangleleft$

Note. If $u_1, \ldots, u_r$ are linearly dependent, with
$$\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_r u_r = 0,$$
and α_1 (say) $\neq 0$ then
$$u_1 = -(\alpha_1^{-1} \alpha_2 u_2 + \cdots + \alpha_1^{-1} \alpha_r u_r).$$
Thus $u_1, \ldots, u_r$ linearly dependent iff one of them is a linear combination of the others.

Definition. A sequence of vectors $u_1, \ldots, u_n$ in a K-vector space M is called a *basis* for M if

(i) $u_1, \ldots, u_n$ are linearly independent;

(ii) $u_1, \ldots, u_n$ generate M.

Definition. If $u_1, \ldots, u_n$ is a basis for a vector space M then for each $x \in M$ we have:
$$x = \alpha^1 u_1 + \cdots + \alpha^n u_n$$
for a sequence of scalars:
$$(\alpha^1, \ldots, \alpha^n),$$
which are called the *coordinates of* x *with respect to the basis* $u_1, \ldots, u_n$.

The coordinates of x are uniquely determined once the basis is chosen because:

$$x = \alpha^1 u_1 + \cdots + \alpha^n u_n = \beta^1 u_1 + \cdots + \beta^n u_n$$

implies:

$$(\alpha^1 - \beta^1)u_1 + \cdots + (\alpha^n - \alpha^n)u_n = 0,$$

and hence

$$\alpha^1 - \beta^1 = 0, \ldots, \alpha^n - \beta^n = 0,$$

by the linear independence of $u_1, \ldots, u_n$. So

$$\alpha^1 = \beta^1, \ldots, \alpha^n = \beta^n.$$

A choice of basis therefore gives a well-defined bijective map:

$$\begin{aligned} M &\to K^n \\ x &\mapsto \text{coordinates of } x, \end{aligned}$$

called the *coordinate map* wrt the given basis.

The following theorem (our first) implies that any two bases for M must have the same number of elements.

Theorem 1.1. *Let M be a K-vector space, $u_1, \ldots, u_n$ be linearly independent in M, and $y_1, \ldots, y_n$ generate M. Then $n \leq r$.*

Proof ▸

$$u_1 = \alpha_1 y_1 + \cdots + \alpha_r y_r$$

(say), since $y_1, \ldots, y_r$ generate M. $\alpha_1, \ldots, \alpha_r$ are not all zero, since $u_1 \neq 0$. Therefore $\alpha_1 \neq 0$ (say). Therefore y_1 is a linear combination of $u_1, y_2, y_3, \ldots, y_r$. Therefore $u_1, y_2, y_3, \ldots, y_r$ generate M. Therefore

$$u_2 = \beta_1 u_1 + \beta_2 y_2 + \beta_3 y_3 + \cdots + \beta_r y_r$$

(say). $\beta_2, \ldots, \beta_r$ are not all zero, since u_1, u_2 are linearly independent. Therefore $\beta_2 \neq 0$ (say). Therefore y_2 is a linear combination of $u_1, u_2, y_3, \ldots, y_r$. Therefore $u_1, u_2, y_3, \ldots, y_r$ generate M.

Continuing in this way, if $n > r$ we get $u_1, \ldots, u_r$ generate M, and hence u_n is a linear combination of $u_1, \ldots, u_r$, which contradicts the linear independence of $u_1, \ldots, u_n$. Therefore $n \leq r$. ◂

Note. If $u_1, \ldots, u_n$ and $y_1, \ldots, y_r$ are two bases for M then $n = r$.

Definition. A vector space M is called *finite-dimensional* if it has a finite basis. The number of elements in a basis is then called the *dimension of* M, denoted by $\dim M$.

Examples:

1. The n vectors:

$$e_1 = (1,0,0,\ldots,0), e_2 = (0,1,0,\ldots,0), \ldots, e_n = (0,0,\ldots,0,1)$$

   form a basis for K^n as a vector-space, called the *usual basis* for K^n.

   Proof of This ▷ We have

$$\begin{aligned} \alpha_1 e_1 + \cdots + \alpha_n e_n &= \alpha_1(1,0,\ldots,0) + \cdots + \alpha_n(0,\ldots,0,1) \\ &= (\alpha_1, \alpha_2, \ldots, \alpha_n). \end{aligned}$$

   Therefore

   (a) $e_1, \ldots, e_n$ generate K^n;

   (b) $\alpha_1 e_1 + \cdots + \alpha_n e_n = 0 \Rightarrow \omega_1 = 0, \ldots, \omega_n = 0$.

   Therefore $\alpha_1, \ldots, \alpha_n$ are linearly independent. ◁

2. The mn matrices:

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

   form a basis for $K^{m\times n}$ as a K-vector space.

3. The functions $\cos\omega x$, $\sin\omega x$ form a basis for the solutions of the equation

$$\frac{d^2u}{dx^2} + \omega^2 u = 0 \quad (\omega \neq 0).$$

4. The functions

$$1, x, x^2, \ldots, x^n$$

   form a basis for the subspace of $C^\infty(\mathbb{R})$ consisting of polynomial functions of degree $\leq n$.

5. $\dim K^n = n$; $\dim K^{m\times n} = mn$. We have:

$$\dim \mathbb{C}^{m\times n} = \begin{cases} mn & \text{as a complex vector space;} \\ 2mn & \text{as a real vector space.} \end{cases}$$

Given any linearly independent set of vectors we can add extra ones to form a basis. Given any generating set of vectors we can discard some to form a basis. More generally:

Theorem 1.2. *Let M be a vector space with a finite generating set (or a vector subspace of such a space). Let Z be a generating set, and let X be a linearly independent subset of Z. Then M has a finite basis Y such that*

$$X \subset Y \subset Z.$$

Proof ▶ Among all the linearly independent subsets of Z which contain X there is one at least

$$Y = \{u_1, \ldots, u_n\},$$

with a maximal number of elements, n (say).

Now if $z \in Z$ then $z, u_1, \ldots, u_n$ are linearly dependent. Therefore there exist scalars $\lambda, \alpha_1, \ldots, \alpha_n$ not all zero such that

$$\lambda z + \alpha_1 u_1 + \cdots + \alpha_n u_n = 0.$$

$\lambda \neq 0$, since $u_1, \ldots, u_n$ are linearly independent. Therefore z is a linear combination of $u_1, \ldots, u_n$.

But Z generates M. Therefore $u_1, \ldots, u_n$ generate M. Therefore $u_1, \ldots, u_n$ form a basis for M. ◀

Chapter 2

Linear Operators 1

2.1 The Definition

Definition. Let M, N be K-vector spaces. A map

$$M \xrightarrow{T} N$$

is called a *linear operator* (or *linear map* or *linear function* or *linear transformation* or *linear homomorphism*) if

(i) $T(x + y) = Tx + Ty$ (group homomorphism);

(ii) $T\alpha x = \alpha Tx$ for all $x, y \in M$, $\alpha \in K$.

A linear operator is called a (*linear*) *isomorphism* if T is bijective. We say that M is *isomorphic to* N if there exists a linear isomorphism

$$M \to N.$$

Note. Geometrically:

(i) means that T preserves parallelograms (see Figure 2.1);

(ii) means that T preserves collinearity (see Figure 2.2).

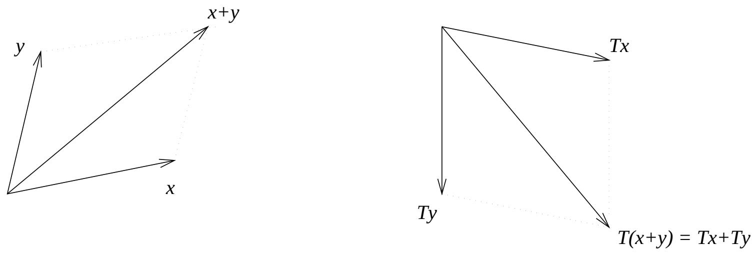


Figure 2.1

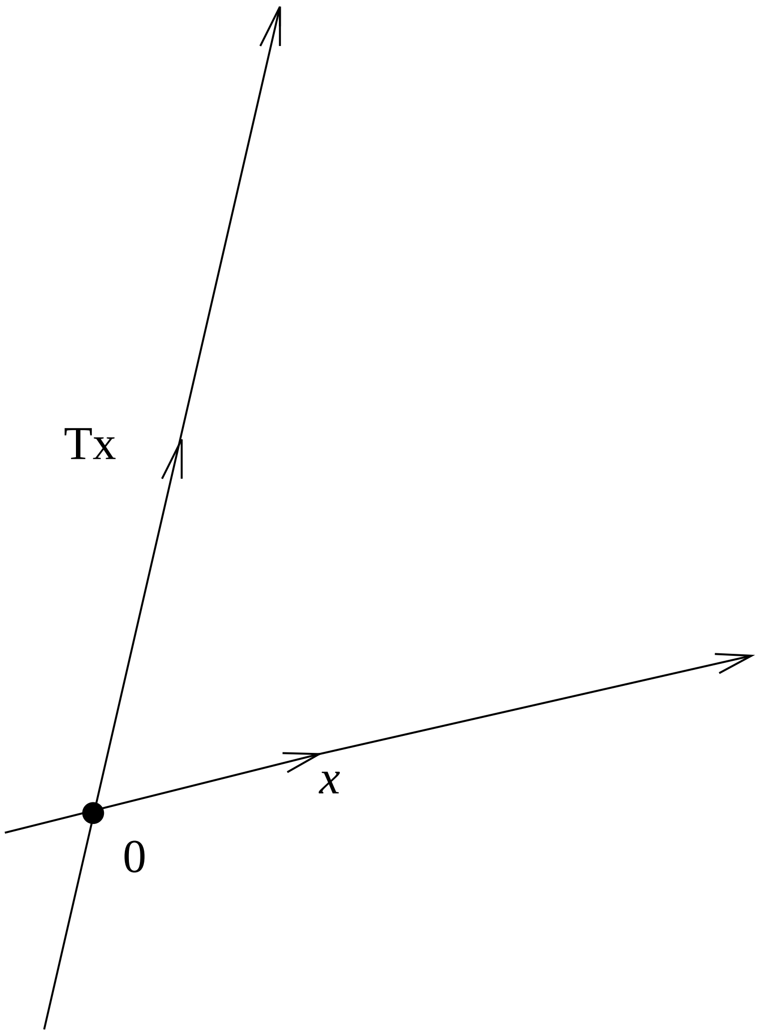


Figure 2.2

Examples:

1. If

$$A = (\alpha^i_j) = \begin{pmatrix} \alpha^1_1 & \dots & \alpha^1_n \\ \vdots & & \vdots \\ \alpha^m_1 & \dots & \alpha^m_n \end{pmatrix} \in K^{m \times n},$$

we denote by

$$K^n \xrightarrow{A} K^m$$

the linear operator given by matrix multiplication by A acting on elements of K^n written as $n \times 1$ columns. Since

$$A(x+y) = Ax + Ay,$$
$$A\alpha x = \alpha A x$$

for matrix multiplication, it follows that A is a linear operator.
E.g.

$$A = \begin{pmatrix} 3 & 7 & 2 \\ -2 & 5 & 1 \end{pmatrix} \in \mathbb{R}^{2\times 3}$$

Now:

$$\mathbb{R}^3 \to \mathbb{R}^2 \;:\; \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \mapsto \begin{pmatrix} 3\alpha + 7\beta + 2\gamma \\ -2\alpha + 5\beta + \gamma \end{pmatrix}.$$

2. Take

$$\frac{d}{dt} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}).$$

Now:

$$\frac{d}{dt}[x(t) + y(t)] = \frac{d}{dt}x(t) + \frac{d}{dt}y(t),$$
$$\frac{d}{dt}cx(t) = c\frac{d}{dt}x(t)$$

for all $c \in \mathbb{R}$. Therefore $\frac{d}{dt}$ is a linear operator.

3. The *Laplacian*

$$\Delta = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2} : C^\infty(\mathbb{R}^3) \to C^\infty(\mathbb{R}^3)$$

is a linear operator.

2.2 Basic Properties of Linear Operators

1. If $M \xrightarrow{T} N$ is a linear operator and $u_1, \ldots, u_r \in M$; $\alpha_1, \ldots, \alpha_r \in K$ then

$$T(\alpha_1 u_1 + \cdots + \alpha_r u_r) = \alpha_1 T u_1 + \cdots + \alpha_r T u_r,$$

i.e.

$$T \sum_{i=1}^{r} \alpha_i u_i = \sum_{i=1}^{r} \alpha_i u_i,$$

i.e. T preserves linear combinations, i.e. T can be moved across summations and scalars.

2. If $M \stackrel{S,T}{\to} N$ are linear operators, if $u_1, \dots, u_m$ generate M, and if $Su_i = Tu_i$ $(i = 1, \dots, m)$ then $S = T$.

   Proof of This ▷ Let $x \in M$. Then $x = \sum_{i=1}^m \alpha_i u_i$ (say). Therefore

   $$Sx = S\sum_{i=1}^m \alpha_i u_i = \sum_{i=1}^m \alpha_i S u_i = \sum_{i=1}^m \alpha_i T u_i = T\sum_{i=1}^m \alpha_i u_i = Tx.$$

   ◁

   Thus two linear operators which agree on a generating set must be equal.

3. Let $u_1, \dots, u_n$ be a basis for M, and $w_1, \dots, w_n$ be arbitrary vectors in N. Then we can define a linear operator

   $$M \xrightarrow{T} N$$

   by

   $$T(\alpha_1 u_1 + \cdots + \alpha_n u_n) = \alpha_1 w_1 + \cdots + \alpha_n u_n.$$

   Thus T is the unique linear operator such that

   $$Tu_i = w_i \quad (i = 1, \dots, m).$$

   We say that T is *defined by* $Tu_i = w_i$, and *extended to* M *by linearity.*

Definition. Let $M \xrightarrow{T} N$ be a linear operator. Then

$$\ker T = \{x \in M : Tx = 0\}$$

is a vector subspace of M, called the *kernel of* M, and

$$\operatorname{im} T = \{Tx : x \in M\}$$

is a vector subspace of N, called the *image of* T. The dimension of $\operatorname{im} T$ is called the *rank of* T,

$$\operatorname{rank} T = \dim \operatorname{im} T.$$

2.3 Examples

1. Consider the matrix operator

$$K^n \xrightarrow{A} K^m,$$

where $A \in K^{m\times n}$,

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \vdots & & & \vdots \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mn} \end{pmatrix}$$

(say).

$$\ker T = \{x = (x_1, \dots, x_n) : Ax = 0\}$$

is the space of solutions of

$$\begin{pmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{m1} & \dots & \alpha_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix},$$

i.e. The space of solutions of the m homogeneous linear equations in n unknowns, whose coefficients are the rows of A:

$$\begin{aligned} \alpha_{11}x_1 + \alpha_{12}x_2 + \dots + \alpha_{1n}x_n &= 0 \\ &\vdots \\ \alpha_{i1}x_1 + \alpha_{i2}x_2 + \dots + \alpha_{in}x_n &= 0 \\ &\vdots \\ \alpha_{m1}x_1 + \alpha_{m2}x_2 + \dots + \alpha_{mn}x_n &= 0 \end{aligned}$$

Number of equations $= m =$ number of rows of $A = \dim K^m$.

Number of unknowns $= n =$ number of columns of $A = \dim K^n$.

We see that $(x_1, x_2, \dots, x_n) \in \ker A$ iff the dot product:

$$(\alpha_{i1}, \alpha_{i2}, \dots, \alpha_{in}).(x_1, \dots, x_n) \quad (i = 1, \dots, m)$$

with each row of A is zero. Therefore

$$\ker A = (\text{row } A)^\perp,$$

where row A is the vector subspace of K^n generated by the m rows of A (see Figure 2.3).

Now row A is unchanged by the following *elementary row operations*:

(i) multiplying a row by a non-zero echelon;

(ii) interchanging rows;

(iii) adding to one row a scalar multiple of another row.

So $\ker A$ is also unchanged by these operations.

To obtain a basis for $\operatorname{row} A$, and from this a basis for $\ker A$, carry out elementary row operations in order to bring the matrix to *row echelon form* (i.e. so that each row begins with more zeros than the previous row).

Example: Let

$$A = \begin{pmatrix} 2 & 1 & -1 & 3 \\ -1 & 1 & 2 & 1 \\ 4 & 0 & -1 & 2 \end{pmatrix} : \mathbb{R}^4 \to \mathbb{R}^3.$$

Now

$$\begin{aligned} A &\sim \begin{pmatrix} 2 & 1 & -1 & 3 \\ 0 & 3 & 3 & 5 \\ 0 & -2 & 1 & -4 \end{pmatrix} \qquad \begin{matrix} 2\,\text{row}\,2 + \text{row}\,1 \\ \text{row}\,3 - 2\,\text{row}\,1 \end{matrix} \\ &\sim \begin{pmatrix} 2 & 1 & -1 & 3 \\ 0 & 3 & 3 & 5 \\ 0 & 0 & 9 & -2 \end{pmatrix} \qquad 3\,\text{row}\,3 + 2\,\text{row}\,2. \end{aligned}$$

Since the new rows are in row echelon form they are linearly independent. Therefore $\operatorname{row} A$ is 3-dimensional, with basis $(2,1,-1,3)$, $(0,3,3,5)$, $(0,0,9,-2)$. Therefore

$$\begin{aligned} (\alpha,\beta,\gamma,\delta) \in \ker A \Leftrightarrow\ & 2\alpha + \beta - \gamma + 3\delta = 0 \\ & 3\beta + 3\gamma + 5\delta = 0 \\ & 9\gamma - 2\delta = 0 \\ \Leftrightarrow\ & \gamma = \tfrac{2}{9}\delta \\ & 3\beta = -3\gamma - 5\delta = -\tfrac{2}{3}\delta - 5\delta = -\tfrac{17}{3}\delta \\ & 2\alpha = -\beta + \gamma - 3\delta = \tfrac{17}{9}\delta + \tfrac{2}{9}\delta - 3\delta = -\tfrac{8}{9}\delta \\ \Leftrightarrow\ & (\alpha,\beta,\gamma,\delta) = (-\tfrac{4}{9}\delta, -\tfrac{17}{9}\delta, \tfrac{2}{9}\delta, \delta) = \tfrac{\delta}{9}(-4,-17,2,9) \end{aligned}$$

Therefore $\ker A$ is 1-dimensional, with basis $(-4,-17,2,9)$.

If

$$A = \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1j} & \dots & \alpha_{1n} \\ \alpha_{21} & \dots & \alpha_{2j} & \dots & \alpha_{2n} \\ \vdots & & \vdots & & \vdots \\ \alpha_{m1} & \dots & \alpha_{mj} & \dots & \alpha_{mn} \end{pmatrix} \in K^{m\times n}$$

then

$$\begin{aligned} Ae_j &= \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1j} & \dots & \alpha_{1n} \\ \alpha_{21} & \dots & \alpha_{2j} & \dots & \alpha_{2n} \\ \vdots & & \vdots & & \vdots \\ \alpha_{m1} & \dots & \alpha_{mj} & \dots & \alpha_{mn} \end{pmatrix} \begin{pmatrix} 0 \\ \cdot \\ 1 \\ \cdot \\ 0 \end{pmatrix} \leftarrow j^{th} \text{ slot} \\ &= \begin{pmatrix} \alpha_{1j} \\ \vdots \\ \alpha_{mj} \end{pmatrix} = j^{th} \text{ column of } A. \end{aligned}$$

Therefore

$$\begin{aligned} \operatorname{im} A &= \{Ax : x \in K^n\} \\ &= \{A(\alpha_1 e_1 + \cdots + \alpha_n e_n) : \alpha_1, \ldots, \alpha_n \in K\} \\ &= \{\alpha_1 Ae_1 + \cdots + \alpha_n Ae_n : \alpha_1, \ldots, \alpha_n \in K\} \\ &= \mathcal{S}(Ae_1, \ldots, Ae_n) \\ &= \text{column space of } A \\ &= \operatorname{col} A, \end{aligned}$$

where $\operatorname{col} A$ is the vector subspace of K^m generated by the n columns of A.

To find a basis for $\operatorname{im} A = \operatorname{col} A$ we carry out elementary column operations on A.

Example: If

$$A = \begin{pmatrix} 2 & 1 & -1 & 3 \\ -1 & 1 & 2 & 1 \\ 4 & 0 & -1 & 2 \end{pmatrix}$$

then

$$A \sim \begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 3 & 3 & 5 \\ 4 & -4 & 2 & -8 \end{pmatrix} \qquad \begin{matrix} 2\,\mathrm{col}\,2 - \mathrm{col}\,1 \\ 2\,\mathrm{col}\,3 + \mathrm{col}\,1 \\ 2\,\mathrm{col}\,4 - 3\,\mathrm{col}\,1 \end{matrix}$$

$$\sim \begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 4 & -4 & 6 & -4 \end{pmatrix} \qquad \begin{matrix} \mathrm{col}\,3 - 2\,\mathrm{col}\,2 \\ 3\,\mathrm{col}\,4 - 5\,\mathrm{col}\,2 \end{matrix}$$

$$\sim \begin{pmatrix} 2 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 4 & -4 & 6 & 0 \end{pmatrix}.$$

Therefore $\mathrm{im}\,A = \mathrm{col}\,A$ has basis $(2, -1, 4),\ (0, 3, -4),\ (0, 0, 6)$. Therefore $\mathrm{rank}\,A = \dim \mathrm{im}\,A = 3$.

2. Let

$$D = \frac{d}{dt} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}) \quad (Dx(t) = \frac{d}{dt}x(t)).$$

(i) Let $\lambda \in \mathbb{R}$ and $D - \lambda$ be the operator

$$(D - \lambda) = \frac{d}{dt}x(t) - \lambda x(t).$$

Then

$$x \in \ker(D - \lambda) \Leftrightarrow (D - \lambda)x = 0 \Leftrightarrow \frac{dx}{dt} = \lambda x \Leftrightarrow x(t) = ce^{\lambda t}.$$

Therefore $\ker(D - \lambda)$ is 1-dimensional, with basis $e^{\lambda t}$.

(ii) To determine $\ker(D - \lambda)^k$ we must solve:

$$(D - \lambda)^k x = 0.$$

Put $x(t) = e^{\lambda t} y(t)$. Then

$$\begin{aligned} (D - \lambda)x &= Dx(t) - \lambda x(t) \\ &= \lambda e^{\lambda t} y(t) + e^{\lambda t} Dy(t) - \lambda e^{\lambda t} y(t) \\ &= e^{\lambda t} Dy(t). \end{aligned}$$

Therefore

$$\begin{aligned} (D - \lambda)^2 x &= e^{\lambda t} D^2 y(t) \\ &\ \vdots \\ (D - \lambda)^k x &= e^{\lambda t} D^k y(t). \end{aligned}$$

Therefore

$$\begin{aligned}(D-\lambda)^k x = 0 &\Leftrightarrow e^{\lambda t} D^k y(t) = 0\\ &\Leftrightarrow D^k y(t) = 0\\ &\Leftrightarrow y(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_{k-1} t^{k-1}\\ &\Leftrightarrow x(t) = (c_0 + c_1 t + \cdots + c_{k-1} t^{k-1}) e^{\lambda t}.\end{aligned}$$

Therefore $\ker(D-\lambda)^k$ is k-dimensional, with basis $e^{\lambda t}, te^{\lambda t}, t^2 e^{\lambda t}, \ldots, t^{k-1} e^{\lambda t}$.

2.4 Properties Continued

Theorem 2.1. *Let $M \xrightarrow{T} N$ be a linear operator, where M is finite dimensional. Let $u_1, \ldots, u_k$ be a basis for $\ker T$, and let $Tw_1, \ldots, Tw_r$ be a basis for $\operatorname{im} T$. Then*

$$u_1, \ldots, u_k, w_1, \ldots, w_r$$

is a basis for M.

Proof ▶ We have two things to show:

(i) *Linear independence*: Let

$$\sum \alpha_i u_i + \sum \beta_j w_j = 0$$

Apply T:

$$0 + \sum \beta_j T w_j = 0.$$

Therefore $\beta_j = 0$ for all j. Therefore $\alpha_i = 0$ for all i.

Therefore $u_1, \ldots, u_k, w_1, \ldots, w_r$ are linearly independent.

(ii) *Generate*: Let $x \in M$. Then

$$Tx = \sum \beta_j T w_j \quad \text{(say)}.$$

Therefore

$$Tx = T \sum \beta_j w_j.$$

Therefore

$$T[x - \sum \beta_j w_j] = 0.$$

Therefore

$$x - \sum \beta_j w_j \in \ker T.$$

Therefore

$$x - \sum \beta_j w_j = \sum \alpha_i u_i \quad \text{(say)}.$$

Therefore

$$x = \sum \alpha_i u_i + \sum \beta_j w_j.$$

Therefore $u_1, \dots, u_k, w_1, \dots, w_r$ generate M. ◄

Corollary 2.1. $\dim \ker T + \dim \operatorname{im} T = \dim M$.

Corollary 2.2. *If* $\dim M = \dim N$ *then*

T *is injective* $\Leftrightarrow \ker T = \{0\} \Leftrightarrow \dim \operatorname{im} T = \dim N \Leftrightarrow T$ *is surjective.*

2.5 Operator Algebra

If M, N are K-vector spaces, we denote by

$$\mathcal{L}(M, N)$$

the set of all linear operators $M \to N$, and we denote by

$$\mathcal{L}(M)$$

the set of all linear operators $M \to M$.

Theorem 2.2. *We have:*

(i) $\mathcal{L}(M, N)$ *is a* K*-vector space, with*

$$(S + T)x = Sx + Tx,$$
$$(\alpha T)x = \alpha(Tx)$$

for all $S, T \in \mathcal{L}(M, N),\ x \in M,\ \alpha \in K$.

(ii) Composition of operators gives a multiplication

$$\left.\begin{array}{rcl} \mathcal{L}(L, M) \times \mathcal{L}(M, N) & \to & \mathcal{L}(L, N) \\ (T, S) & \mapsto & ST \end{array}\right\} \quad L \xrightarrow{T} M \xrightarrow{S} N,$$

with

$$(ST)x = S(Tx) \quad \text{for all } x \in L,$$

which satisfies

(a) $(RS)T = R(ST)$,

(b) $R(S+T) = RS + RT$,

(c) $(R+S)T = RT + ST$,

(d) $(\alpha S)T = \alpha(ST) = S(\alpha T)$,

provided each is well-defined.

Proof ▶ Straight forward verification. ◀

Corollary 2.3. $\mathcal{L}(M)$ *is*

(i) *a* K*-vector space:* $S+T,\ \alpha S$;

(ii) *a ring:* $S+T,\ ST$;

(iii) $(\alpha S)T = \alpha(ST) = S(\alpha T) :\quad \alpha S,\ ST$,

i.e. $\mathcal{L}(M)$ *is a* K*-algebra.*

2.6 Isomorphisms of $\mathcal{L}(M,N)$ with $K^{m\times n}$

Definition. Let $u_1,\ldots,u_n$ be a basis for M, and let $w_1,\ldots,w_m$ be a basis for N. Let $M \xrightarrow{T} N$. Put Then we have:

$$\begin{aligned}
Tu_1 &= \alpha_1^1 w_1 + \alpha_1^2 w_2 + \cdots + \alpha_1^i w_i + \cdots + \alpha_1^m w_m,\\
&\ \vdots\\
Tu_j &= \alpha_j^1 w_1 + \alpha_j^2 w_2 + \cdots + \alpha_j^i w_i + \cdots + \alpha_j^m w_m,\\
&\ \vdots\\
Tu_n &= \alpha_n^1 w_1 + \alpha_n^2 w_2 + \cdots + \alpha_n^i w_i + \cdots + \alpha_n^m w_m,
\end{aligned}$$

(say) where:

$$A = (\alpha_j^i) = \begin{pmatrix} \alpha_1^1 & \alpha_2^1 & \ldots & \alpha_j^1 & \ldots & \alpha_n^1 \\ \vdots & & & \vdots & & \vdots \\ \alpha_1^i & \ldots & \ldots & \alpha_j^i & \ldots & \alpha_n^i \\ \vdots & & & \vdots & & \vdots \\ \alpha_1^m & \ldots & \ldots & \alpha_j^m & \ldots & \alpha_n^m \end{pmatrix} \in K^{m\times n}.$$

Note. The coordinates of Tu_j form the j^{th} column of A - NOTE THE TRANSPOSE! We call A the *matrix of* T wrt the bases $u_1,\ldots,u_n$ for M and $w_1,\ldots,w_m$ for N,

$$Tu_j = \sum_{i=1}^{m} \alpha_j^i \omega_i.$$

Theorem 2.3. *$\mathcal{L}(M,N) \to K^{m\times n}$ is a linear isomorphism where $T \to$ matrix of T w.r.t. basis $u_1, \cdots, u_n; \omega_1, cdots, \omega_m$.*

Proof ▶ Let T have matrix $A = (\alpha^i_j)$, and let S have matrix $B = (\beta^i_j)$. Then

$$(T+S)u_j = Tu_j + Su_j = \sum_{i=1}^m \alpha^i_j w_i + \sum_{i=1}^m \beta^i_j w_i = \sum_{i=1}^m (\alpha^i_j + \beta^i_j) w_i.$$

Therefore $T+S$ has matrix $(\alpha^i_j + \beta^i_j) = A + B$. Also

$$(\lambda T)u_j = \lambda(Tu_j) = \lambda \sum_{i=1}^m \alpha^i_j w_i = \sum_{i=1}^m \lambda \alpha^i_j w_i.$$

Therefore λT has matrix $(\lambda \alpha^i_j) = \lambda A$. ◀

Corollary 2.4. $\dim \mathcal{L}(M,N) = \dim M . \dim N$.

Theorem 2.4. *If $L \xrightarrow{T} M$ has matrix $A = (\alpha^i_j)$ wrt basis $v_1, \ldots, v_p, u_1, \ldots, u_n$, and $M \xrightarrow{S} N$ has matrix $B = (\beta^i_j)$ wrt basis $u_1, \ldots, u_n, w_1, \ldots, w_m$ then $L \xrightarrow{ST} N$ has basis*

$$BA = \left(\sum_{k=1}^n \beta^i_k \alpha^k_j \right) = (\gamma^i_j)$$

(say), wrt basis $v_1, \ldots, v_p, w_1, \ldots, w_m$.

Proof ▶

$$\begin{aligned}(ST)v_j = S(Tv_j) = S\left(\sum_{k=1}^n \alpha^k_j u_k\right) &= \sum_{k=1}^n \alpha^k_j S u_k \\ &= \sum_{k=1}^n \alpha^k_j \sum_{i=1}^m \beta^i_k w_i = \sum_{i=1}^m \left(\sum_{k=1}^n \beta^i_k \alpha^k_j\right) w_i = \sum_{i=1}^m \gamma^i_j w_i.\end{aligned}$$

◀

Corollary 2.5. *If $\dim M = n$ then each choice of basis $u_1, \ldots, u_n$ of M defines an isomorphism of K-algebras:*

$$\mathcal{L}(M) \to K^m \ : \ T \mapsto \text{ matrix of } T \text{ wrt } u_1, \ldots, u_n.$$

Note. If $M \xrightarrow{T} M$ has matrix $A = (\alpha^i_j)$ wrt basis $u_1, \ldots, u_n$ then

(i) $Tu_j = \sum_{i=1}^n \alpha^i_j u_i$, by definition;

(ii) the elements of the j^{th} column of A are the coordinates of Tu_j;

(iii) $\lambda_0 1 + \lambda_1 T + \lambda_2 T^2 + \cdots + \lambda_r T^r$ has matrix $\alpha_0 I + \alpha_1 A + \cdots + \alpha_r A^r$;

(iv) T^{-1} has matrix A^{-1},

since we have an algebra isomorphism.

Theorem 2.5. *Let $M \xrightarrow{T} N$ have matrix $A = (\alpha^i_j)$ wrt bases $u_1, \ldots, u_n$ for M and $w_1, \ldots, w_m$ for N. Let x have coordinates*

$$X = (\xi^i) = \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^n \end{pmatrix}$$

wrt $u_1, \ldots, u_n$. Then Tx has coordinates

$$AX = \left(\sum_{i=1}^{m} \alpha^i_j \xi^j \right)$$

wrt $w_1, \ldots, w_m$.

Proof ▶

$$Tx = T\left(\sum_{j=1}^{n} \xi^j u_j \right) = \sum_{j=1}^{n} \xi^i T u_j = \sum_{j=1}^{n} \xi^j \sum_{i=1}^{m} \alpha^i_j w_i = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} \alpha^i_j \xi^j \right) w_i,$$

as required. ◀

Note. We have thus a commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{T} & N \\ \downarrow & & \downarrow \\ K^n & \xrightarrow{A} & K^m \end{array} \quad : \quad \begin{array}{ccc} x & \stackrel{T}{\mapsto} & Tx \\ \downarrow & & \downarrow \\ x \text{ coord.} & \stackrel{A}{\mapsto} & Tx \text{ coord} \end{array}$$

Chapter 3

Changing Basis and Einstein Convention

Definition. If $u_1, \overset{\text{old}}{\ldots}, u_n$ and $w_1, \overset{\text{new}}{\ldots}, w_n$ are two bases for M then we have:

$$\begin{aligned}
u_1 &= p_1^1 w_1 + p_1^2 w_2 + \cdots + p_1^n w_n \\
&\vdots \\
u_j &= p_j^1 w_1 + p_j^2 w_2 + \cdots + p_j^n w_n \\
&\vdots \\
u_n &= p_n^1 w_1 + p_n^2 w_2 + \cdots + p_n^n w_n
\end{aligned}$$

(say). Put

$$P = (p_j^i) = \begin{pmatrix} p_1^1 & \ldots & p_j^1 & \ldots & p_n^1 \\ p_1^2 & \ldots & p_j^2 & \ldots & p_n^2 \\ \vdots & & \vdots & & \vdots \\ p_1^n & \ldots & p_j^2 & \ldots & p_n^n \end{pmatrix}$$

Note. The new coordinates of the old basis vector u_j form the j^{th} *column of* P - NOTE THE TRANSPOSE! We call P the *transition matrix* from the (old) basis $u_1, \ldots, u_n$ to the (new) basis $w_1, \ldots, w_n$:

$$u_j = \sum_{i=1}^{n} p_j^i w_i.$$

Theorem 3.1. *If x has old coordinates*

$$X = (\xi^i) = \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^n \end{pmatrix}$$

then x has new coordinates

$$PX = \sum_{j=1}^{n} (p^i_j \xi^j) = (\eta^i)$$

(say).

Proof ▶

$$x = \sum_{j=1}^{n} \xi^j u_j = \sum_{j=1}^{n} \xi^j \sum_{i=1}^{n} p^i_j w_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} p^i_j \xi^j \right) w_i = \sum_{i=1}^{n} \eta^i w_i.$$

◀

We shall often use the *Einstein summation convention* (*s.c.*) when dealing with basis and coordinates in a fixed n-dimensional vector space M. Repeated indices (one up, one down) are summed from 1 to n (*contraction* of repeated indices). Non-repeated indices may take each value 1 to n.

Example:

- α^i denotes

$$\begin{pmatrix} \alpha^1 \\ \vdots \\ \alpha^n \end{pmatrix} \quad \text{(column matrix; upper index labels the row).}$$

- α_i denotes

$$(\alpha_1, \ldots, \alpha_n) \quad \text{(row matrix; lower index labels the column).}$$

- α^i_j denotes

$$\begin{pmatrix} \alpha^1_1 & \ldots & \alpha^1_n \\ \vdots & & \vdots \\ \alpha^n_1 & \ldots & \alpha^n_n \end{pmatrix} \quad \text{(square matrix).}$$

- u_i denotes $u_1, \ldots, u_n$ (basis).
- $\alpha^i u_i$ denotes $\alpha^1 u_1 + \cdots + \alpha^n u_n$.
- $\alpha^i \beta_i$ denotes $\alpha^1 \beta_1 + \cdots + \alpha^n \beta_n$ (dot product).

- $\alpha^i_k \beta^k_j$ denotes AB (matrix product).

Also

$$Tu_j = \alpha^i_j u_i \quad (\alpha^i_j \text{ matrix of operator } T)$$

and

$$u_j = p^i_j w_i \quad (p^i_j \text{ transition matrix from } u_i \text{ to } w_i).$$

If x has components ξ^i wrt u_i then Tx has components $\alpha^i_j \xi^i$ wrt u_i. If x has components ξ^j wrt u_i then x has components $p^i_j \xi^j$ wrt w_i.

- δ^i_j denotes the unit matrix

$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{pmatrix}.$$

- If $Q = P^{-1}$ then (q^i_j) denotes Q (inverse matrix) and

$$q^i_k p^k_j = \delta^i_j = p^i_k q^k_j.$$

Theorem 3.2. *Let $M \xrightarrow{T} N$ have matrix A wrt basis $u_1, \dots, u_n$. Let P be the transition matrix to (new) basis $w_1, \dots, w_n$. Then T has (new) matrix*

$$PAP^{-1}$$

wrt $w_1, \dots, w_n$.

Proof ▶ Let $P = (p^i_j),\ A = (\alpha^i_j),\ P^{-1} = Q = (q^i_j)$. Then

$$Tu_j = \alpha^i_j u_i; \quad u_j = p^i_j w_i; \quad w_j = q^i_j u_i.$$

Therefore

$$Tw_j = Tq^l_j u_l = q^l_j T u_l = q^l_j \alpha^k_l u_k = q^l_j \alpha^k_l p^i_k w_i = \underbrace{p^i_k \alpha^k_l q^l_j}_{PAP^{-1}} w_i,$$

as required. ◀

Chapter 4

Linear Forms and Duality

4.1 Linear Forms

Definition. Fix M a K-vector space. A scalar valued linear function

$$f : M \to K$$

is called a *linear form* on M.

If f is a linear form on M, and x is a vector in M, we write

$$\langle f, x \rangle$$

to denote the value of f on x. This notation has the advantage of treating f and x in a symmetrised way:

(i) $\langle f, x + y \rangle = \langle f, x \rangle + \langle f, y \rangle$,

(ii) $\langle f + g, x \rangle = \langle f, x \rangle + \langle g, x \rangle$,

(iii) $\langle \alpha f, x \rangle = \alpha \langle f, x \rangle = \langle f, \alpha x \rangle$,

(iv) $\left\langle \sum_{i=1}^{r} \alpha_i f^i, \sum_{j=1}^{s} \beta^j x_j \right\rangle = \sum_{i=1}^{r} \sum_{j=1}^{s} \alpha_i \beta^j \langle f^i, x_j \rangle$.

If M is finite dimensional, with basis $u_1, \ldots, u_n$, then each $x \in M$ can be written uniquely as

$$x = \alpha^1 u_1 + \cdots + \alpha^n u_n = \sum_{i=1}^{n} \alpha^i u_i = \alpha^i u_i.$$

We write

$$\langle u^i, x \rangle = \alpha^i$$

to denote the i^{th} coordinate of x wrt basis $u_1, \ldots, u_n$. We have:

$$\begin{aligned} \langle u^i, x + y \rangle &= \langle u^i, x \rangle + \langle u^i, y \rangle, \\ \langle u^i, \alpha x \rangle &= \alpha \langle u^i, x \rangle. \end{aligned}$$

Thus u^i is a linear form on M, called the i^{th} *coordinate function wrt basis* $u_1, \ldots, u_n$. We have:

1. $\langle u^i, u_j \rangle = \left\{ \begin{array}{ll} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{array} \right\} = \delta^i_j$ (*Kronecker delta*);
2. $x = \sum_{i=1}^n \langle u^i, x \rangle u_i$ for all $x \in M$;
3. $\langle \alpha_1 u^1 + \cdots + \alpha_n u^n, \beta^1 u_1 + \cdots + \beta^n u_n \rangle = \alpha_1 \beta^1 + \cdots + \alpha_n \beta^n = \alpha_i \beta^i$ (*dot product*).

Theorem 4.1. *If $u_1, \ldots, u_n$ is a basis for M then the coordinate functions $u^1, \ldots, u^n$ form a basis for the space M^* of linear forms on M (called the* dual space *of M), called the* dual basis*, and*

$$f = \sum_{i=1}^n \langle f, u_i \rangle u^i \quad \textit{for each } f \in M^*.$$

Proof ▶ We have to show that $u^1, \ldots, u^n$ generate M, and are linearly independent.

(i) *Generate*: Let $f \in M^*$; $\langle f, u_j \rangle = \beta_j$ (say). Then

$$\left\langle \sum_{i=1}^n \beta_i u^i, u_j \right\rangle = \sum_{i=1}^n \beta_i \langle u^i, u_j \rangle = \sum_{i=1}^n \beta_i \delta^i_j = \beta_j = \langle f, u_j \rangle.$$

Therefore $\sum_{i=1}^n \beta_i u^i$ and f are linear forms on M which agree on the basis vectors $u_1, \ldots, u_n$. Therefore

$$f = \sum_{i=1}^n \beta_i u^i = \sum_{i=1}^n \langle f, u_i \rangle u^i.$$

(ii) *Linear independence*: Let $\sum_{i=1}^n \beta_i u^i = 0$. Then

$$\left\langle \sum_{i=1}^n \beta_i u^i, u_j \right\rangle = 0$$

for all $j = 1, \dots, n$. Therefore

$$\sum_{i=1}^{n} \beta_i \delta_j^i = 0$$

for all $j = 1, \dots, n$. Therefore $\beta_j = 0$ for all $j = 1, \dots, n$. Therefore $u^1, \dots, u^n$ are linearly independent. ◂

Corollary 4.1. $\dim M^* = \dim M$.

Note. We denote by x, y, z the coordinate function on K^3 wrt basis e_1, e_2, e_3, and we denote by $x^1, \dots, x^n$ the coordinate function on K^n wrt basis $e_1, \dots, e_n$. These coordinates are called the *usual coordinates*.

4.2 Duality

Let M be finite dimensional, with dual space M^*. If $x \in M$ and $f \in M^*$ then

(i) f is a linear form on M whose value on x is $\langle f, x \rangle$;

(ii) we identify x with the linear form on M^* whose value on f is $\langle f, x \rangle$:

$$f = \langle f, \cdot \rangle,$$
$$x = \langle \cdot, x \rangle.$$

What we are doing is identifying M with the dual of M^*, by means of the linear isomorphism:

$$M \to M^{**}$$
$$x \mapsto \langle \cdot, x \rangle.$$

This is a linear map, and is bijective because:

(i) $\dim M^{**} = \dim M^* = \dim M$,

(ii) $\langle \cdot, x \rangle = 0 \Rightarrow \langle u^i, x \rangle = 0$ for all $x \Rightarrow x = 0$. So the map is injective (kernel $= \{0\}$), and hence by (i) surjective.

If $u_1, \dots, u_n$ is a basis for M, and $u^1, \dots, u^n$ the dual basis for M^* then

$$\langle u^i, u_j \rangle = \delta_j^i$$

shows that $u_1, \dots, u_n$ is the basis dual to $u^1, \dots, u^n$.

The identification of vectors $x \in M$ as linear forms on M^* is called *duality*. A basis $u^1, \dots, u^n$ for M^* is called a *linear coordinate system* on M, and consists of coordinate functions wrt its dual basis $u_1, \dots, u_n$.

4.3 Systems of Linear Equations

Definition. If $f^1, \ldots, f^k$ are linear forms on M then we consider the vector subspace of M on which

$$f^1 = 0, \ldots, f^k = 0 \quad (*).$$

Any vector in this subspace is called a *solution* of the equations $(*)$. Thus $x \in M$ is a solution iff

$$\langle f^1, x\rangle = 0, \ldots, \langle f^k, x\rangle = 0.$$

The set of solutions is called the *solution space* of the *system of k homogeneous equations* $(*)$. The dimension of the space $\mathcal{S}(f^1, \ldots, f^k)$ generated by $f^1, \ldots, f^k$ is called the *rank* (number of linearly independent equations) of the system of equations.

In particular, if $u^1, \ldots, u^n$ is a linear coordinate system on M then we can write the equations as:

$$\begin{aligned} f^1 &\equiv \beta^1_1 u^1 + \cdots + \beta^1_n u^n = 0 \\ &\vdots \\ f^k &\equiv \beta^k_1 u^1 + \cdots + \beta^k_n u^n = 0 \end{aligned}$$

The coordinate map $M^* \to K^n$ maps

$$\begin{aligned} f^1 &\mapsto (\beta^1_1, \ldots, \beta^1_n) \\ &\vdots \\ f^k &\mapsto (\beta^k_1, \ldots, \beta^k_n). \end{aligned}$$

Thus it maps $\mathcal{S}(f^1, \ldots, f^k)$ isomorphically onto the row space of $B = (\beta^i_j)$. Therefore

$$\text{rank of system} = \text{dimension of row space of } B = \dim \operatorname{row} B.$$

Example: The equations

$$\begin{aligned} 3x - 4y + 2z &= 0, \\ 2x + 7y + 3z &= 0, \end{aligned}$$

where x, y, z are the usual coordinates on $\mathbb{R}^3$, have

$$\text{rank} = \dim \operatorname{row} \begin{pmatrix} 3 & -4 & 2 \\ 2 & 7 & 3 \end{pmatrix} = 2.$$

Theorem 4.2. *A system of k homogeneous linear equations of rank r on an n-dimensional vector space M has a solution space of dimension $n - r$.*

Proof ▶ Let

$$f^1 = 0, \ldots, f^k = 0$$

be the system of equations. Let $u^1, \ldots, u^r$ be a basis for $\mathcal{S}(f^1, \ldots, f^k)$. Extend to a basis $\underbrace{u^1, \ldots, u^r}_{\text{equations}}, u^{r+1}, \ldots, u^n$ for M^*. Let $u_1, \ldots, u_r, \underbrace{u_{r+1}, \ldots, u_n}_{\text{solutions}}$ be the dual basis of M. Then

$$\begin{aligned} x = \alpha^1 u_1 + \cdots + \alpha^r u_r + \alpha^{r+1} u_{r+1} + \cdots + \alpha^n u_n &\in \text{ solution space} \\ \Leftrightarrow \alpha^1 = \langle u^1, x \rangle = 0, \ldots, \alpha^r = \langle u^r, x \rangle = 0& \\ \Leftrightarrow x = \alpha^{r+1} u_{r+1} + \cdots + \alpha^n u_n.& \end{aligned}$$

Therefore $u_{r+1}, \ldots, u_n$ is a basis for the solution space. Therefore solution space has dimension $n - r$. ◀

Theorem 4.3. *Let $B \in K^{k \times n}$, where K is a field. Then*

$$\dim \operatorname{row} B = \dim \operatorname{col} B \ (= \operatorname{rank} B).$$

Proof ▶ Consider the k homogeneous linear equations on K^n with coefficients $B = (\beta^i_j)$:

$$\begin{aligned} \beta^1_1 x^1 + \cdots + \beta^1_n x^n &= 0 \\ &\vdots \\ \beta^k_1 x^1 + \cdots + \beta^k_n x^n &= 0. \end{aligned}$$

Now

$$\begin{aligned} n - \dim \operatorname{row} B &= n - \text{rank of equations} \\ &= \text{dimension of solution space} \\ &= \dim \ker B \\ &= n - \dim \operatorname{im} B \\ &= n - \dim \operatorname{col} B. \end{aligned}$$

Therefore $\dim \operatorname{col} B = \dim \operatorname{row} B$. ◀

Chapter 5

Tensors

5.1 The Definition

Definition. Let M be a finite dimensional vector space over a field K, let M^* be the dual space, and let $\dim M = n$. A *tensor* over M is a function of the form

$$T : M_1 \times M_2 \times \cdots \times M_k \to K,$$

where each $M_i = M$ or M^* $(i = 1, \ldots, k)$, and which is linear in each variable (*multilinear*).

Two tensors S, T are said to be of the *same type* if they are defined on the same set $M_1 \times \cdots \times M_k$.

Example: A tensor of type

$$T : M \times M^* \times M \to K$$

is a scalar valued function $T(x, f, y)$ of three variables (x a vector, f a linear form, y a vector) such that

$$\begin{aligned} T(\alpha x + \beta y, f, z) &= \alpha T(x, f, z) + \beta T(y, f, z) && \text{linear in } 1^{st} \text{ variable,} \\ T(x, \alpha f + \beta g, z) &= \alpha T(x, f, z) + \beta T(x, g, z) && \text{linear in } 2^{nd} \text{ variable,} \\ T(x, f, \alpha y + \beta z) &= \alpha T(x, f, z) + \beta T(x, f, z) && \text{linear in } 3^{rd} \text{ variable.} \end{aligned}$$

If u_i is a basis for M, and u^i is the dual basis for M^* then the array of n^3 scalars

$$\alpha_i{}^j{}_k = T(u_i, u^j, u_k)$$

are called the *components* of T.

If x, f, y have components ξ^i, η_j, ρ^k respectively then

$$T(x, f, y) = T(\xi^i u_i, \eta_j u^j, \rho^k u_k) = \xi^i \eta_j \rho^k T(u_i, u^j, u_k) = \xi^i \eta_j \rho^k \alpha_i{}^j{}_k$$

(using summation notation), i.e. the components of T contracted by the components of x, f, y.

The set of all tensors over M of a given type form a K-vector space if we define

$$\begin{aligned}(S+T)(x_1, \ldots, x_k) &= S(x_1, \ldots, x_k) + T(x_1, \ldots, x_k),\\ (\lambda T)(x_1, \ldots, x_k) &= \lambda(T(x_1, \ldots, x_k)).\end{aligned}$$

The vector space of all tensors of type

$$M \times M^* \times M \to K$$

(say) has dimension n^3, since $T \mapsto T(u_i, u^j, u_k)$ (components of T) maps it isomorphically onto K^{n^3}.

Definition. If $S : M_1 \times \cdots \times M_k \to K$ and $T : M_{k+1} \times \cdots \times M_l \to K$ are tensors over M then we define their *tensor product* $S \otimes T$ to be the tensor:

$$S \otimes T : M_1 \times \cdots \times M_k \times M_{k+1} \times \cdots \times M_l \to K,$$

where

$$S \otimes T(x_1, \ldots, x_l) = S(x_1, \ldots, x_k)T(x_{k+1}, \ldots, x_l).$$

Example: If S has components $\alpha_i{}^j{}_k$, and T has components β^{rs} then $S \otimes T$ has components $\alpha_i{}^j{}_k \beta^{rs}$, because

$$S \otimes T(u_i, u^j, u_k, u^r, u^s) = S(u_i, u^j, u_k)T(u^r, u^s).$$

Tensors satisfy algebraic laws such as:

(i) $R \otimes (S + T) = R \otimes S + R \otimes T$,

(ii) $(\lambda R) \otimes S = \lambda(R \otimes S) = R \otimes (\lambda S)$,

(iii) $(R \otimes S) \otimes T = R \otimes (S \otimes T)$.

But

$$S \otimes T \neq T \otimes S$$

in general. To prove those we look at components wrt a basis, and note that

$$\alpha^i{}_{jk}(\beta^r{}_s + \gamma^r{}_s) = \alpha^i{}_{jk}\beta^r{}_s + \alpha^i{}_{jk}\gamma^r{}_s,$$

for example, but

$$\alpha^i \beta^j \neq \beta^j \alpha^i$$

in general.

5.2 Contraction

Definition. Let $T : M_1 \times \cdots \times M_r \times \cdots \times M_s \times \cdots \times M_k \to K$ be a tensor, with

$$M_r = M^*, \quad M_s = M$$

(say). Then we can *contract* the r^{th} *index* of T *with* the s^{th} *index* to get a new tensor

$$S : M_1 \times \cdots \times \overset{\text{omit}}{M_r} \times \cdots \times \overset{\text{omit}}{M_s} \times \cdots \times M_k \to K$$

defined by

$$S(x_1, x_2, \ldots, x_{k-2}) = T(x_1, \ldots, \underset{r^{th}\text{ slot}}{u^i}, \ldots, \underset{s^{th}\text{ slot}}{u_i}, \ldots, x_{k-2}),$$

where u_i is a basis for M.

To show that S is well-defined we need:

Theorem 5.1. *The definition of contraction is independent of the choice of basis.*

Proof ► Put

$$R(f, x) = T(x_1, x_2, \ldots, f, \ldots, x, \ldots, x_{k-2}).$$

Then if u_i, w_i are bases:

$$R(w^i, w_i) = R(p^i_k u^k, q^l_i u_l) = p^i_k q^l_i R(u^k, u_l) = \delta^l_k R(u^k, u_l) = R(u^k, u_k),$$

as required. ◄

Example: If T has components $\alpha^i{}_{jk}{}^{lm}$ wrt basis u_i then contraction of the 2^{nd} and 4^{th} indices gives a tensor with components

$$\beta^i{}_k{}^m = T(u^i, u_j, u_k, u^j, u^m) = \alpha^i{}_{jk}{}^{jm}.$$

Thus when we contract we eliminate one upper (*contravariant*) index and one lower (*covariant*) index.

5.3 Examples

A vector $x \in M$ is a tensor:

$$x : M^* \to K$$

with components $\alpha^i = \langle u^i, x\rangle$ (one contravariant index).

A linear form $f \in M^*$ is a tensor:

$$f : M \to K$$

with components $\alpha_i = \langle f, u_i\rangle$ (one covariant index).

A tensor with two covariant indices:

$$T : M \times M \to K,$$

with $T(u_i, u_j) = \alpha_{ij}$, is called a *bilinear form* or *scalar product*.

Example: The dot product

$$K^n \times K^n \to K$$
$$((\alpha^1, \dots, \alpha^n), (\beta^1, \dots, \beta^n)) \mapsto \alpha^1\beta^1 + \cdots + \alpha^n\beta^n$$

is a bilinear form on K^n.

If $M \xrightarrow{T} M$ is a linear operator, we shall identify it with the tensor:

$$T : M^* \times M \to K$$

by

$$T(f, x) = \langle f, Tx\rangle.$$

This tensor has components

$$\alpha^i{}_j = T(u^i, u_j) = \langle u^i, Tu_j\rangle = \text{ matrix of linear operator } T$$

(one contravariant index, one covariant index).

Note (The Transformation Law). Let p^i_j be the transition matrix from basis u_i to basis w_i, with inverse matrix q^j_i. Let T be a tensor $M \times M^* \times M \to K$ (say). Then

$$\overbrace{T(w_i, w^j, w_k)}^{\text{new comps.}} = T(q^r_i u_r, p^j_s u^s, q^t_k u_t) = q^r_i p^j_s q^t_k \overbrace{T(u_r, u^s, u_t)}^{\text{old comps.}},$$

i.e. Upper indices contract with p, lower indices contract with q.

5.4 Bases of Tensor Spaces

Let $M \times M^* \times M \to K$ $(*)$ (say) be a tensor with components $\alpha_i{}^j{}_k$ wrt basis u_i. Then the tensor:

$$\alpha_i{}^j{}_k \, u^i \otimes u_j \otimes u^k \quad (**)$$

is of the same type as T, and has components

$$\begin{aligned} \alpha_i{}^j{}_k u^i \otimes u_j \otimes u^k[u_r, u^s, u_t] &= \alpha_i{}^j{}_k \langle u^i, u_r\rangle \langle u^s, u_j\rangle \langle u^k, u_t\rangle \\ &= \alpha_i{}^j{}_k \delta^i_r \delta^s_j \delta^k_t \\ &= \alpha_r{}^s{}_t. \end{aligned}$$

Therefore $(**)$ has the same components as T. Therefore

$$T = \alpha_i{}^j{}_k \, u^i \otimes u_j \otimes u^k.$$

Therefore $u^i \otimes u_j \otimes u^k$ is a basis for the n^3-dimensional space of all tensors of type $(*)$.

Chapter 6

Vector Fields

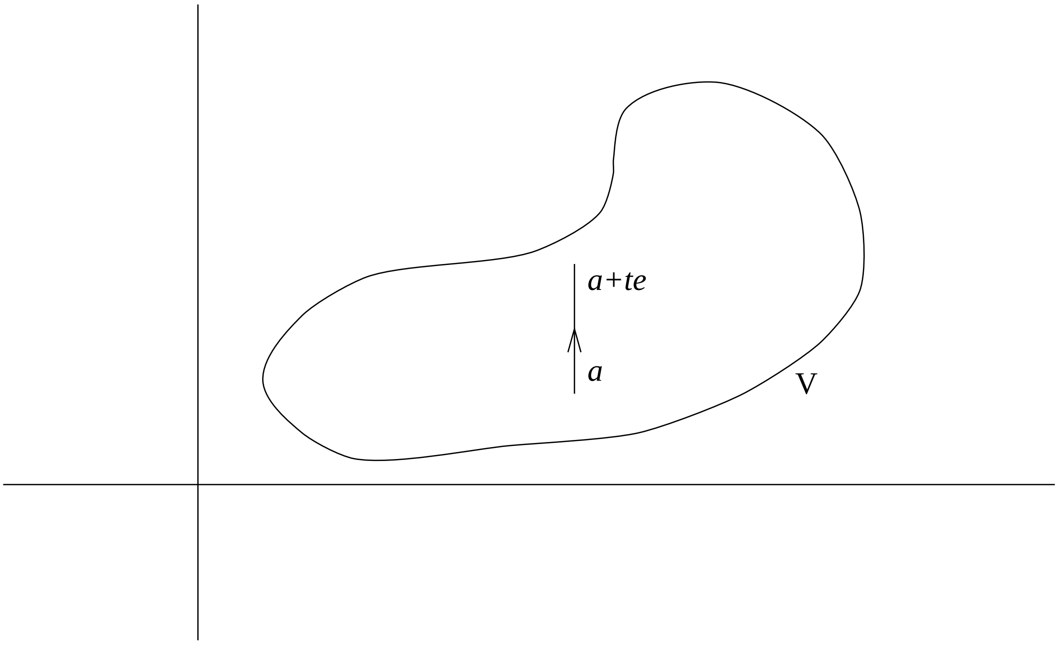
6.1 The Definition

Let V be an open subset of $\mathbb{R}^n$. Let $x^1, \ldots, x^n$ be the usual coordinate functions on $\mathbb{R}^n$. Let $V \xrightarrow{f} \mathbb{R}$. If $a = (a_1, \ldots, a_n) \in V$ then we define the *partial derivative of f wrt i^{th} variable at a*:

$$\begin{aligned}
\frac{\partial f}{\partial x^i}(a) &= \lim_{t \to 0} \frac{f(a_1, \ldots, a_i + t, \ldots, a_n) - f(a_1, \ldots, a_i, \ldots, a_n)}{t} \\
&= \lim_{t \to 0} \frac{f(a + te_i) - f(a)}{t} \\
&= \frac{d}{dt} f(a + te_i)|_{t=0}
\end{aligned}$$

(see Figure 6.1). If it exists for each $a \in V$ then we have:

$$\frac{\partial f}{\partial x^i} : V \to \mathbb{R}.$$



$Figure: 6.1$

Note that

$$\frac{\partial x^i}{\partial x^j} = \delta^i_j.$$

If all repeated partial derivatives of all orders:

$$\frac{\partial^r f}{\partial x^{i_1} \cdots \partial x^{i_r}} = \frac{\partial}{\partial x^{i_1}} \cdots \frac{\partial}{\partial x^{i_r}} f : V \to \mathbb{R}$$

exist we call f C^∞. We denote by $C^\infty(V)$ the space of all C^∞ functions $V \to \mathbb{R}$. $C^\infty(V)$ is an $\mathbb{R}$-algebra:

(i) $(f + g)(x) = f(x) + g(x)$,

(ii) $(fg)(x) = f(x)g(x)$,

(iii) $(\alpha f)(x) = \alpha(f(x))$.

Each sequence $\alpha^1, \ldots, \alpha^n$ of elements of $C^\infty(V)$ defines a linear operator

$$v = \alpha^1 \frac{\partial}{\partial x^1} + \cdots + \alpha^n \frac{\partial}{\partial x^n}$$

on $C^\infty(V)$, where

$$(vf)(x) = \alpha^1(x)\frac{\partial f}{\partial x^1}(x) + \cdots + \alpha^n(x)\frac{\partial f}{\partial x^n}(x).$$

Such an operator

$$v : C^\infty(V) \to C^\infty(V)$$

is called a (contravariant) *vector field on V*.

Now for each fixed a we denote by

$$\frac{\partial}{\partial x^i_a}$$

the operator given by:

$$\frac{\partial}{\partial x^i_a} f = \frac{\partial f}{\partial x^i}(a).$$

Thus $\frac{\partial}{\partial x^i_a}$ acts on any function f which is defined and C^1 on an open set containing a. We define the linear combination $\sum_{i=1}^n \alpha^i \frac{\partial}{\partial x^i_a}$ by

$$\left(\alpha^1 \frac{\partial}{\partial x^i_a} + \cdots + \alpha^n \frac{\partial}{\partial x^n_a}\right) f = \alpha^1 \frac{\partial f}{\partial x^1}(a) + \cdots + \alpha^n \frac{\partial f}{\partial x^n}(a).$$

The set of linear combinations

$$\left\{\alpha^1 \frac{\partial}{\partial x^1_a} + \cdots + \alpha^n \frac{\partial}{\partial x^n_a} : \alpha^1, \ldots, \alpha^n \in \mathbb{R}\right\}$$

is called the *tangent space to $\mathbb{R}^n$ at a*, denoted $T_a\mathbb{R}^n$. Thus $T_a\mathbb{R}^n$ is a real n-dimensional vector space, with basis

$$\frac{\partial}{\partial x^1_a}, \ldots, \frac{\partial}{\partial x^n_a}.$$

The operators $\frac{\partial}{\partial x^i_a}$ are linearly independent, since

$$\alpha^1 \frac{\partial}{\partial x^1_a} + \cdots + \alpha^n \frac{\partial}{\partial x^n_a} = 0 \Rightarrow \left(\alpha^1 \frac{\partial}{\partial x^1_a} + \cdots + \alpha^n \frac{\partial}{\partial x^n_a}\right) x^i = 0 \Rightarrow \alpha^i = 0,$$

since $\frac{\partial x^i}{\partial x^j}(a) = \delta^i_j$.

If $v = \alpha^1 \frac{\partial}{\partial x^1} + \cdots + \alpha^n \frac{\partial}{\partial x^n}$ ($\alpha^i \in C^\infty(V)$) is a vector field on V then we have (see Figure 6.2), for each $x \in V$ a tangent vector

$$v_x = \alpha^1(x)\frac{\partial}{\partial x^1_x} + \cdots + \alpha^n(x)\frac{\partial}{\partial x^n_x} \in T_x\mathbb{R}^n.$$

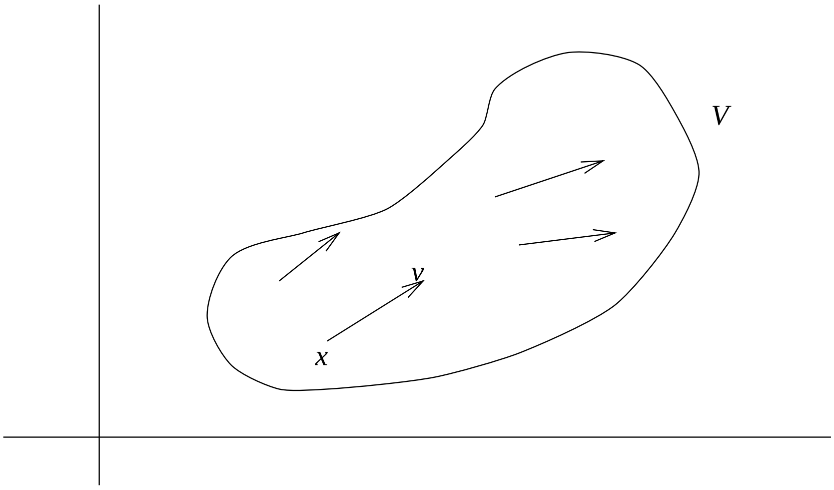


Figure 6.2

We call v_x the *value of* v *at* x, and note that

$$\begin{aligned} v_x f &= \left(\alpha^1(x)\frac{\partial}{\partial x^1_x} + \cdots + \alpha^n(x)\frac{\partial}{\partial x^n_x}\right) f \\ &= \alpha^1(x)\frac{\partial f}{\partial x^1}(x) + \cdots + \alpha^n(x)\frac{\partial f}{\partial x^n}(x) \\ &= (vf)(x) \end{aligned}$$

for all $x \in V$. Thus v is determined by its values $\{v_x : x \in V\}$, and vice versa. Thus a contravariant vector field is a function on V

$$x \mapsto v_x,$$

which maps to each point $x \in V$ a tangent vector $v_x \in T_x\mathbb{R}^n$.

6.2 Velocity Vectors

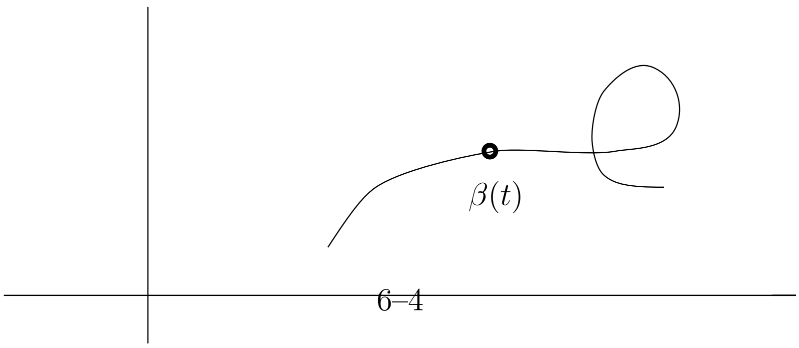


Figure 6.3

Let $\beta(t) = (\beta^1(t), \ldots, \beta^n(t))$ be a sequence of real valued C^∞ functions defined on an open subset of $\mathbb{R}$. Thus $\beta = (\beta^1, \ldots, \beta^n)$ is a *curve* in $\mathbb{R}^n$ (see Figure 6.3). If f is a C^∞ real-valued function on an open set in $\mathbb{R}^n$ containing $\beta(t)$ then the rate of change of f along the curve β at parameter t is

$$\begin{aligned}
\frac{d}{dt} f(\beta(t)) &= \frac{d}{dt} f(\beta^1(t), \ldots, \beta^n(t)) \\
&= \frac{\partial f}{\partial x^1}(\beta(t)) \frac{d}{dt}\beta^1(t) + \cdots + \frac{\partial f}{\partial x^n}(\beta(t)) \frac{d}{dt}\beta^n(t) \quad \text{(by the chain rule)} \\
&= \left[\frac{d}{dt}\beta^1(t) \frac{\partial}{\partial x^1_{\beta(t)}} + \cdots + \frac{d}{dt}\beta^n(t) \frac{\partial}{\partial x^n_{\beta(t)}} \right] f \\
&= \dot\beta(t) f,
\end{aligned}$$

where

$$\dot\beta(t) = \frac{d}{dt}\beta^1(t) \frac{\partial}{\partial x^1_{\beta(t)}} + \cdots + \frac{d}{dt}\beta^n(t) \frac{\partial}{\partial x^n_{\beta(t)}} \in T_{\beta(t)}\mathbb{R}^n$$

is called the *velocity vector* of β at t.

We note that if $\beta(t)$ has coordinates

$$\beta^i(t) = x^i(\beta(t))$$

then $\dot\beta(t)$ has components

$$\begin{aligned}
\frac{d}{dt}\beta^i(t) &= \frac{d}{dt} x^i(\beta(t)) \\
&= \text{rate of change of } x^i \text{ along } \beta \text{ at } t \text{ wrt basis} \tfrac{\partial}{\partial x^1_{\beta(t)}}, \ldots, \tfrac{\partial}{\partial x^n_{\beta(t)}}.
\end{aligned}$$

In particular, if $\alpha = (\alpha^1, \ldots, \alpha^n) \in \mathbb{R}^n$ and $a = (a^1, \ldots, a^n) \in \mathbb{R}^n$ then the straight line through a (see Figure 6.4) in the direction of α:

$$(a^1 + t\alpha^1, \ldots, a^n + t\alpha^n)$$

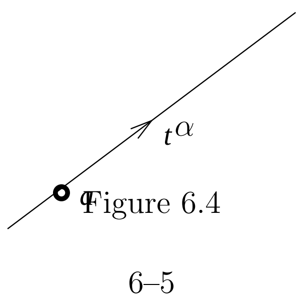


Figure 6.4

has velocity vector at $t = 0$:

$$\alpha^1 \frac{\partial}{\partial x_a^1} + \cdots + \alpha^n \frac{\partial}{\partial x_a^n} \in T_a\mathbb{R}^n.$$

Thus each tangent vector is a velocity vector.

6.3 Differentials

Definition. If $a \in \mathbb{R}^n$, and f is a C^∞ function on an open neighbourhood of a then the *differential of f at a*, denoted

$$df_a,$$

is the linear form on $T_a\mathbb{R}^n$ defined by

$$\langle df_a, \dot{\beta}(t)\rangle = \frac{d}{dt} f(\beta(t)) = \dot{\beta}(t) f$$

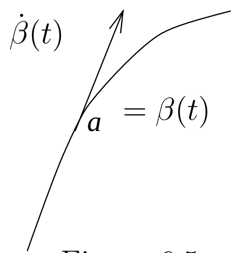


Figure 6.5

for any velocity vector $\dot{\beta}(t)$, such that $\beta(t) = a$.

Thus

(i) $\langle df_{\beta(t)}, \dot{\beta}(t)\rangle$ = rate of change of f along β at t (see Figure 6.5),

(ii) $\langle df_a, v\rangle = vf$ (for all $v \in T_a\mathbb{R}^n$) = rate of change of f along v.

Theorem 6.1. *$dx_a^i, \ldots, dx_a^n$ is the basis of $T_a\mathbb{R}^{n*}$ dual to the basis $\frac{\partial}{\partial x_a^1}, \ldots, \frac{\partial}{\partial x_a^n}$ for $T_a\mathbb{R}^n$.*

Proof ▶

$$\left\langle dx_a^i, \frac{\partial}{\partial x_a^j}\right\rangle = \frac{\partial x^i}{\partial x^j}(a) = \delta_j^i,$$

as required. ◀

Definition. If V is open in $\mathbb{R}^n$ then a *covariant vector field* ω on V is a function on V:

$$\omega : x \mapsto \omega_x \in T_x\mathbb{R}^{n*}.$$

The covariant vector fields on V can be added:

$$(\omega + \eta)_x = \omega_x + \eta_x,$$

and multiplied by elements of $C^\infty(V)$:

$$(f\omega)_x = f(x)\omega_x.$$

Each covariant vector field ω on V can be written uniquely as

$$\omega_x = \beta_1(x)dx_x^1 + \cdots + \beta_n(x)dx_x^n.$$

Thus

$$\omega = \beta_1 dx^1 + \cdots + \beta_n dx^n$$

(we confine ourselves to $\beta_i \in C^\infty(V)$).

If $f \in C^\infty(V)$ then the covariant vector field

$$df : x \mapsto df_x$$

is called the *differential of* f. Thus we have:

- contravariant vector fields:

$$v = \alpha^1 \frac{\partial}{\partial x^1} + \cdots + \alpha^n \frac{\partial}{\partial x^n}, \quad \alpha^i \in C^\infty(V);$$

- covariant vector fields:

$$\omega = \beta_1 dx^1 + \cdots + \beta_n dx^n, \quad \beta \in C^\infty(V);$$

and more general *tensor fields*, e.g.

$$S = \alpha_i{}^j{}_k \, dx^i \otimes \frac{\partial}{\partial x^j} \otimes dx^k, \quad \alpha_i{}^j{}_k \in C^\infty(V),$$

a function on V whose value at x is

$$S_x = \alpha_i{}^j{}_k(x) dx_x^i \otimes \frac{\partial}{\partial x_x^j} \otimes dx_x^k,$$

a tensor over $T_x\mathbb{R}^n$.

We can add, multiply and contract tensor fields pointwise (carrying out the operation at each point $x \in V$). For example:

(i) $(R+S)_x = R_x + S_x$,

(ii) $(R \otimes S)_x = R_x \otimes S_x$,

(iii) $(\text{contracted } S)_x = \text{contracted } (S_x)$,

(iv) $(fS)_x = f(x)S_x \quad f \in C^\infty(V)$.

Contracting the covariant vector field $\omega = \beta_1 dx^1 + \cdots + \beta_n dx^n$ with the contravariant vector field $v = \alpha^1 \frac{\partial}{\partial x^1} + \cdots + \alpha^n \frac{\partial}{\partial x^n}$ gives the scalar field

$$\langle \omega, v \rangle = \beta_1 \alpha^1 + \cdots + \beta_n \alpha^n.$$

In particular, if $f \in C^\infty(V)$ has differential df then the scalar field

$$\langle df, v \rangle = vf$$

is the *rate of change of* f *along* v.

If $\omega = \beta_1 dx^1 + \cdots + \beta_n dx^n$ then

$$\beta_i = i^{th} \text{ component of } \omega = \left\langle \omega, \frac{\partial}{\partial x^i} \right\rangle.$$

In particular:

$$i^{th} \text{ compoment of } df = \left\langle df, \frac{\partial}{\partial x^i} \right\rangle = \frac{\partial f}{\partial x^i}.$$

Therefore

$$df = \frac{\partial f}{\partial x^1} dx^1 + \cdots + \frac{\partial f}{\partial x^n} dx^n \quad \text{Chain Rule,}$$

$$\text{rate of change of } f = \frac{\partial f}{\partial x^1} . \text{rate of change of } x^1 + \cdots + \frac{\partial f}{\partial x^n} . \text{rate of change of } x^n.$$

6.4 Transformation Law

A sequence

$$y = (y^1, \ldots, y^n) \quad (y^i \in C^\infty(V))$$

is called a (C^∞) *coordinate system on* V if

$$\begin{aligned} V &\to W \\ x &\mapsto y(x) = (y^1(x), \ldots, y^n(x)) \end{aligned}$$

maps V homeomorphically onto an open set W in $\mathbb{R}^n$, and if

$$x^i = F^i(y^1, \dots, y^n),$$

where $F^i \in C^\infty(W)$.

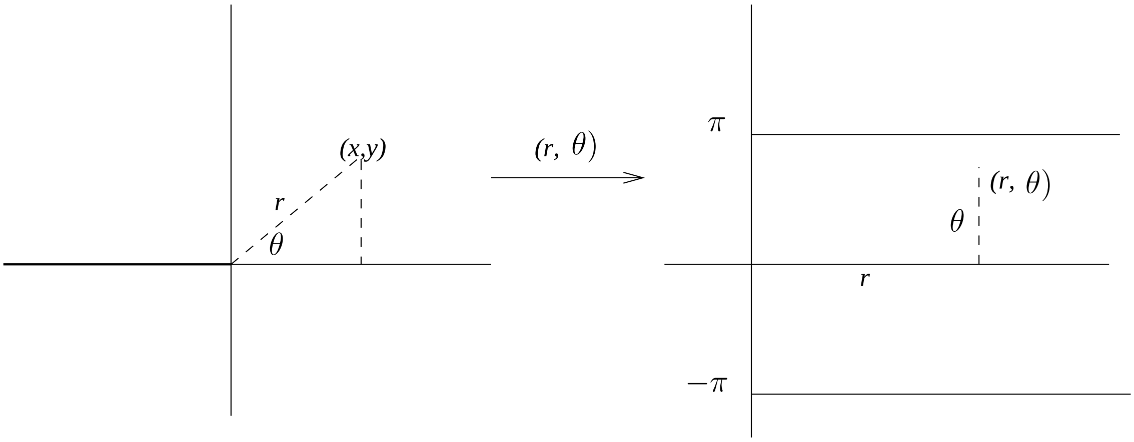


Figure 6.6

Example: (r, θ) is a C^∞ coordinate system on $\{(x, y) : y \text{ or } x > 0\}$ (see Figure 6.6), where $r = \sqrt{x^2 + y^2}$, θ unique solution of $x = r\cos\theta, y = r\sin\theta$ $(-\pi < \theta < \pi)$.

If $a \in V$, and β is the parametrised curve – the curve along which all y^j $(j \neq i)$ are constant, and y^i varies by t – such that

$$y(\beta(t)) = y(a) + te_i$$

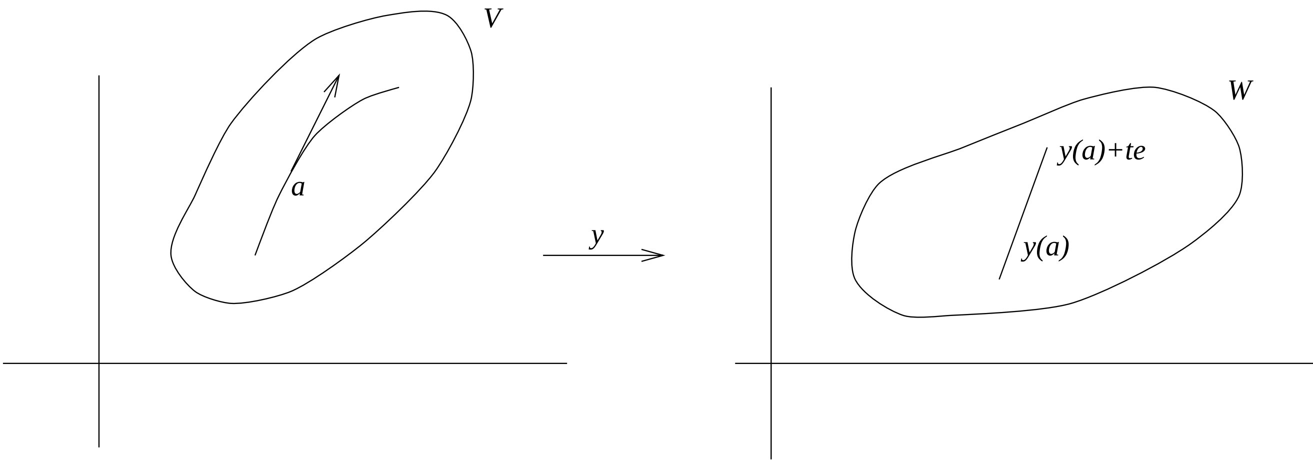


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Figure 6.7

(see Figure 6.7) then the velocity vector of β at $t = 0$ is denoted:

$$\frac{\partial}{\partial y^i_a}$$

Thus if f is C^∞ in a neighbourhood of a then

$$\frac{\partial f}{\partial y^i}(a) = \frac{\partial}{\partial y^i_a} f = \frac{d}{dt} f(\beta(t))|_{t=0} = \text{rate of change of } f \text{ along the curve } \beta.$$

If we write f as a function of $y^1, \ldots, y^n$:

$$f = F(y^1, \ldots, y^n)$$

(say), then

$$\frac{\partial f}{\partial y^i}(a) = \frac{d}{dt} f(\beta(t))|_{t=0} = \frac{d}{dt} F(y(\beta(t)))|_{t=0} = \frac{d}{dt} F(y(a)+te_i)|_{t=0} = \frac{\partial F}{\partial x^i}(y(a)),$$

i.e. to calculate $\frac{\partial f}{\partial y^i}(a)$ write f as a function F of $y^1, \ldots, y^n$, and calculate $\frac{\partial F}{\partial x^i}$ (partial derivative of F wrt i^{th} slot):

$$\frac{\partial f}{\partial y^i} = \frac{\partial F}{\partial x^i}(y^1, \ldots, y^n).$$

Now if β is any parametrised curve at a, with $\beta(t) = a$ (see Figure 6.8), then

$$\begin{aligned}
\langle df_a, \dot{\beta}(t) \rangle &= \frac{d}{dt} f(\beta(t)) \\
&= \frac{d}{dt} F(y^1(\beta(t)), \ldots, y^n(\beta(t))) \\
&= \sum_{i=1}^n \frac{\partial F}{\partial x^i}(y^1(\beta(t)), \ldots, y^n(\beta(t))) \frac{d}{dt} y^i(\beta(t)) \\
&= \sum_{i=1}^n \frac{\partial f}{\partial y^i}(\beta(t)) \langle dy^i_a, \dot{\beta}(t) \rangle
\end{aligned}$$

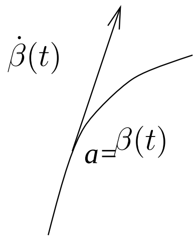

Figure 6.8

Therefore

$$df_a = \sum_{i=1}^{n} \frac{\partial f}{\partial y^i}(a) dy_a^i.$$

Therefore

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial y^i} dy^i.$$

The operators

$$\frac{\partial}{\partial y_a^1}, \ldots, \frac{\partial}{\partial y_a^n}$$

are linearly independent, since $\frac{\partial}{\partial y_a^i} y^i = \delta_j^i$. Therefore these operators form a basis for $T_a\mathbb{R}^n$, with dual basis

$$dy_a^1, \ldots, dy_a^n,$$

since $\langle dy_a^i, \frac{\partial}{\partial y_a^j}\rangle = \frac{\partial y^i}{\partial y^j}(a) = \delta_j^i$.

If $z^1, \ldots, z^n$ is a C^∞ coordinate system on W then on $V \cap W$:

$$dz^i = \sum_{i=1}^{n} \frac{\partial z^i}{\partial y^j} dy^j.$$

Therefore $\frac{\partial z^i}{\partial y^j}$ is the transition matrix from basis $\frac{\partial}{\partial y^i}$ to basis $\frac{\partial}{\partial z^i}$. Therefore

$$\frac{\partial}{\partial y^j} = \sum_{i=1}^{n} \frac{\partial z^i}{\partial y^j} \frac{\partial}{\partial z^i}$$

on $V \cap W$.

If (say) $g = g_{ij} dy^i \otimes dy^j$ is a tensor field on V, with component g_{ij} wrt coordinates y^i, then

$$g = g_{ij} \left(\frac{\partial y^i}{\partial z^k} dz^k\right) \otimes \left(\frac{\partial y^j}{\partial z^l} dz^l\right) = \frac{\partial y^i}{\partial z^k} \frac{\partial y^j}{\partial z^l} g_{ij} dz^k \otimes dz^l,$$

using s.c., and therefore g has component

$$\frac{\partial y^i}{\partial z^k}\frac{\partial y^j}{\partial z^l} g_{ij}$$

wrt coordinates z^i.

Example: On $\mathbb{R}^n$:

(i) usual coordinates x, y;

(ii) polar coordinates r, θ.

$$x = r\cos\theta, \quad y = r\sin\theta.$$

So

$$\begin{aligned} dx &= \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta = \cos\theta\, dr - r\sin\theta\, d\theta, \\ dy &= \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta = \sin\theta\, dr + r\cos\theta\, d\theta. \end{aligned}$$

The matrix

$$\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

is the transition matrix from r, θ to x, y:

$$\begin{aligned} \frac{\partial}{\partial r} &= \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}, \\ \frac{\partial}{\partial \theta} &= \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}. \end{aligned}$$

Chapter 7

Scalar Products

7.1 The Definition

Definition. A tensor of type $M \times M \to K$ is called a *scalar product* or (*bilinear form*) (i.e. two lower indices).

Example: The dot product $K^n \times K^n \to K$. Writing X, Y as $n \times 1$ columns:

$$\begin{aligned} ((\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n)) &\mapsto \alpha_1\beta_1 + \cdots + \alpha_n\beta_n \\ (X, Y) &\mapsto X^t Y. \end{aligned}$$

7.2 Properties of Scalar Products

1. If $(\cdot|\cdot)$ is a scalar product on M with components $G = (g_{ij})$ wrt basis u_i, if x has components $X = (\phi^i)$ and y has components $Y = (\nu^i)$ ($g_{ij} = (u_i|u_j)$ and $(\cdot|\cdot) = g_{ij}u^i \otimes u^j$) then

$$\begin{aligned} (x|y) &= (\phi^i u_i | \nu^j u_j) \\ &= \phi^i \nu^j (u_i|u_j) \\ &= g_{ij}\phi^i\nu^j \\ &= \begin{pmatrix} \phi^1 & \dots & \phi^n \end{pmatrix} \begin{pmatrix} g_{11} & \dots & g_{1n} \\ \vdots & & \vdots \\ g_{n1} & \dots & g_{nn} \end{pmatrix} \begin{pmatrix} \nu^1 \\ \vdots \\ \nu^n \end{pmatrix} \\ &= X^t G Y. \end{aligned}$$

   Note. The dot product has matrix I wrt e_i, since $e_i.e_j = \delta^i_j$.

2. If $P = (p^i_j)$ is the transition matrix to new basis w_i then new matrix of $(\cdot|\cdot)$ is $Q^t G Q$, where $Q = P^{-1}$.

   Proof of This ▷ As a tensor with two lower indices, new components of $(\cdot|\cdot)$ are:
   $$q^k_i q^l_j g_{kl} = q^k_i g_{kl} q^l_j = Q^t G Q.$$
   Check:
   $$(PX)^t Q^t G Q(Y) = X^t P^t Q^t G Q Y = X^t G Y.$$
   ◁

3. $(\cdot|\cdot)$ is called *symmetric* if
   $$(x|y) = (y|x)$$
   for all x, y. This is equivalent to G being a symmetric matrix $G^t = G$:
   $$g_{ij} = (u_i|u_j) = (u_j|u_i) = g_{ji}.$$
   A symmetric scalar product defines an *associated quadratic form*
   $$F : M \to K$$
   by
   $$\begin{aligned} F(x) &= (x|x) \\ &= X^t G X \\ &= \begin{pmatrix} \xi^1 & \dots & \xi^n \end{pmatrix} \begin{pmatrix} g_{11} & \dots & g_{1n} \\ \vdots & & \vdots \\ g_{n1} & \dots & g_{nn} \end{pmatrix} \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^n \end{pmatrix} \\ &= g_{ij}\xi^i\xi^j, \end{aligned}$$
   i.e.
   $$F = \begin{pmatrix} u^1 & \dots & u^n \end{pmatrix} \begin{pmatrix} g_{11} & \dots & g_{1n} \\ \vdots & & \vdots \\ g_{n1} & \dots & g_{nn} \end{pmatrix} \begin{pmatrix} u^1 \\ \vdots \\ u^n \end{pmatrix} = g_{ij} u^i u^j.$$
   $u^i u^j$ is a product of linear forms, and is a function:
   $$(u^i u^j)(x) = u^i(x) u^j(x).$$

Example: If x, y, z are coordinate functions on M then

$$F = \begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 3 & 2 & 3 \\ 2 & -7 & -1 \\ 3 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$
$$= 3x^2 - 7y^2 + 2z^2 + 4xy + 6xz - 2yz.$$

(Thus quadratic form $\equiv$ homogeneous 2^{nd} degree polynomial).

The quadratic form F determines the symmetric scalar product $(\cdot|\cdot)$ uniquely because:

$$(x+y|x+y) = (x|x) + (x|y) + (y|x) + (y|y),$$
$$2(x|y) = F(x+y) - F(x) - F(y) \quad (\text{ if } 1+1 \neq 0),$$

and $g_{ij} = (u_i|u_j)$ are called the *components of* F wrt u_i.

Definition. $(\cdot|\cdot)$ is called *non-singular* if

$$(x|y) = 0 \text{ for all } y \in M \Rightarrow x = 0,$$

i.e.

$$X^t G Y = 0 \text{ for all } Y \in K^n \Rightarrow X = 0,$$

i.e.

$$X^t G = 0 \Rightarrow X = 0,$$

i.e.

$$\det G \neq 0.$$

Definition. A tensor field $(\cdot|\cdot)$ with two lower indices on an open set $V \subset \mathbb{R}^n$:

$$(\cdot|\cdot) = g_{ij} dy^i \otimes dy^j$$

(say), y^i coordinates on V, is called a *metric tensor* if

$$(\cdot|\cdot)_x$$

is a symmetric non-singular scalar product on $T_x\mathbb{R}^n$ for each $x \in V$, i.e.

$$g_{ij} = g_{ji} \quad \text{and} \quad \det g_{ij} \text{ nowhere zero.}$$

The associated field ds^2 of quadratic forms:

$$ds^2 = g_{ij} dy^i dy^j$$

is called the *line-element* associated with the metric tensor.

Example: On $\mathbb{R}^n$ the *usual metric tensor*

$$dx \otimes dx + dy \otimes dy,$$

with line element $ds^2 = (dx)^2 + (dy)^2$, has components

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

wrt coordinates x, y.

If

$$v = v^1 \frac{\partial}{\partial x} + v^2 \frac{\partial}{\partial y}, \quad w = w^1 \frac{\partial}{\partial x} + w^2 \frac{\partial}{\partial y}$$

then

$$(v|w) = \begin{pmatrix} v^1 & v^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} = v^1 w^1 + v^2 w^2 \quad \text{(dot product)}$$

$$ds^2[v] = (v|v) = (v^1)^2 + (v^2)^2 = \|v\|^2 \quad \text{(Euclidean norm).}$$

If r, θ are polar coordinates:

$$x = r \cos\theta, \quad y = r \sin\theta,$$

then

$$\begin{aligned} dx &= \cos\theta \, dr - r \sin\theta \, d\theta, \\ dy &= \sin\theta \, dr + r \cos\theta \, d\theta \end{aligned}$$

and

$$\begin{aligned} ds^2 &= (dx)^2 + (dy)^2 \\ &= (\cos\theta \, dr - r \sin\theta \, d\theta)^2 + (\sin\theta \, dr + r \cos\theta \, d\theta)^2 \\ &= (dr)^2 + r^2 (d\theta)^2 \end{aligned}$$

has components

$$\begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

wrt coordinates r, θ.

If

$$v = \alpha^1 \frac{\partial}{\partial r} + \alpha^2 \frac{\partial}{\partial \theta}, \quad w = \beta^1 \frac{\partial}{\partial r} + \beta^2 \frac{\partial}{\partial \theta}$$

then

$$\begin{aligned} (v|w) &= \alpha^1 \beta^1 + r^2 \alpha^2 \beta^2, \\ \|v\|^2 &= (\alpha^1)^2 + r^2 (\alpha^2)^2. \end{aligned}$$

7.3 Raising and Lowering Indices

Definition. Let M be a finite dimensional vector space with a fixed non-singular symmetric scalar product $(\cdot|\cdot)$. If $x \in M$ is a vector (one upper index), we associate with it

$$\tilde{x} \in M^*,$$

a linear form (one lower index) defined by:

$$\langle \tilde{x}, y \rangle = (x|y) \quad \text{for all } y \in M.$$

We call the operation

$$\begin{aligned} M &\to M^* \\ x &\mapsto \tilde{x} \end{aligned}$$

lowering the index. Thus

$$\tilde{x} \equiv (x|\cdot) \equiv \text{ `take scalar product with } x\text{'}.$$

If $x = \alpha^i u_i$ has components α^i then $\tilde{x}$ has components

$$\alpha_j = \langle \tilde{x}, u_j \rangle = (x|u_j) = (\alpha^i u_i|u_j) = \alpha^i (u_i|u_j) = \alpha^i g_{ij}.$$

Since $(\cdot|\cdot)$ is non-singular, g_{ij} is invertible, with inverse g^{ij} (say), and we have

$$\alpha^j = \alpha_i g^{ij}.$$

Thus

$$\begin{aligned} M &\to M^* \\ x &\mapsto \tilde{x} \end{aligned}$$

is a linear isomorphism, with inverse

$$\underset{\sim}{f} \leftarrow f$$

(say), called *raising the index.* So

$$\begin{aligned} x &= \alpha^i u_i = \underset{\sim}{f}, \\ \tilde{x} &= \alpha_i u^i = f \end{aligned}$$

and

$$(x|y) = (\underset{\sim}{f}|y) = \langle f, y \rangle = \langle \tilde{x}, y \rangle.$$

To lower: contract with g_{ij} $\quad (\alpha_j = \alpha^i g_{ij})$.

To raise: contract with g^{ij} $\quad (\alpha^j = \alpha_i g^{ij})$.

Let $M \xrightarrow{T} M$ be a linear operator and $(\cdot|\cdot)$ be symmetric. The matrix of T is:

$$\alpha^i{}_j = \langle u^i, Tu_j \rangle,$$

one up, one down mixed components of T.

$$\alpha_{ij} = (u_i|Tu_j),$$

two down covariant components of T.

$$\alpha_{ij} = (u_i|\alpha^k{}_j u_k) = (u_i|u_k)\alpha^k{}_j = g_{ik}\alpha^k{}_j$$

(lower by contraction with g_{ij}). Therefore

$$\alpha^i{}_j = g^{ik}\alpha_{kj}$$

(raise by contraction with g^{ij}).

If we take the covariant components α_{ij}, and raise the *second* index we get

$$\alpha_i{}^j = \alpha_{ik} g^{kj}.$$

α_{ij} are the components of the tensor B (two lower indices) defined by:

$$B(x, y) = (x|Ty),$$

since

$$B(u_i, u_j) = (u_i|Tu_j) = \alpha_{ij}.$$

$\alpha_j{}^i$ are the components of an operator T^* (one upper index, one lower index) defined by:

$$(T^*x|y) = (x|Ty),$$

since T^* has components

$$\gamma_{ij} = (u_i|T^*u_j) = (T^*u_j|u_i) = (u_j|Tu_i) = \alpha_{ji},$$

and therefore T^* has mixed components:

$$\gamma^i{}_j = g^{ik}\gamma_{kj} = \alpha_{jk}g^{ki} = \alpha_j{}^i.$$

T^* is called the *adjoint* of operator T.

7.4 Orthogonality and Diagonal Matrix

Definition. If $(\cdot|\cdot)$ is a scalar product on M and

$$(x|y) = 0,$$

we say that x is *orthogonal to* y wrt $(\cdot|\cdot)$.

If N is a vector subspace of M, we write

$$N^{\perp} = \{x \in M : (x|y) = 0 \text{ for all } y \in N\},$$

and call it the *orthogonal complement of* N wrt $(\cdot|\cdot)$ (see Figure 7.1).

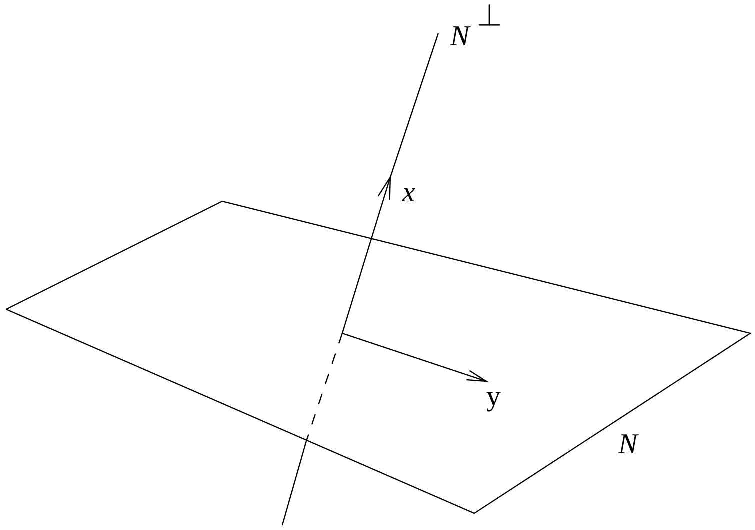

Figure 7.1

We denote by $(\cdot|\cdot)_N$ the scalar product on N defined by

$$(x|y)_N = (x|y) \quad \text{for all } x, y \in N,$$

and call it the *restriction of* $(\cdot|\cdot)$ *to* N.

Definition. Let $N_1, \ldots, N_k$ be vector subspaces of a vector space M. Then we write

$$N_1 + \cdots + N_k = \{x_1 + \cdots + x_k : x_1 \in N_1, \ldots, x_k \in N_k\},$$

and call it the *sum* of $N_1, \ldots, N_k$. Thus $M = N_1 + \cdots + N_k$ iff each $x \in M$ can be written as a sum

$$x = x_1 + \cdots + x_k, \quad x_i \in N_i.$$

We call M a *direct sum* of $N_1, \ldots, N_k$, and write

$$M = N_1 \oplus \cdots \oplus N_k$$

if for each $x \in M$ there exists *unique* $(x_1, \ldots, x_k)$ (for example, see Figure 7.2) such that

$$x = x_1 + \cdots + x_k \quad \text{and} \quad x_i \in N_i.$$

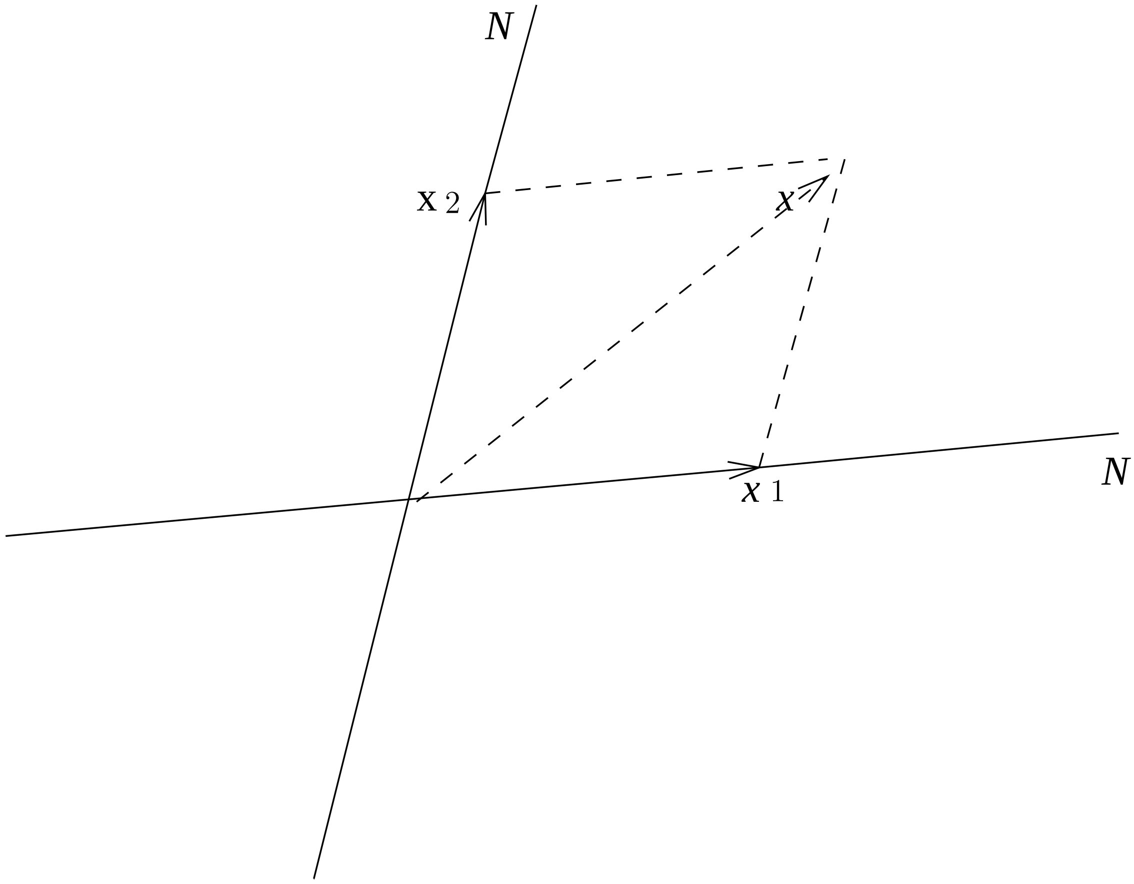

Figure 7.2

Theorem 7.1. *Let $(\cdot|\cdot)$ be a scalar product on M. Let N be a finite-dimensional vector subspace such that $(\cdot|\cdot)_N$ is non-singular. Then*

$$M = N \oplus N^{\perp}.$$

Proof ▶ Let $x \in M$ (see Figure 7.3). Define $f \in N^*$ by

$$\langle f, y \rangle = (x|y)$$

for all $y \in N$.

Since $(\cdot|\cdot)_N$ is non-singular we can raise the index of f, and get a *unique* vector $z \in N$ such that

$$\langle f, y \rangle = (z|y)$$

for all $y \in N$, i.e.

$$(x|y) = (z|y)$$

for all $y \in N$, i.e.

$$(x - z|y) = 0$$

for all $y \in N$, i.e.

$$x - z \in N^\perp,$$

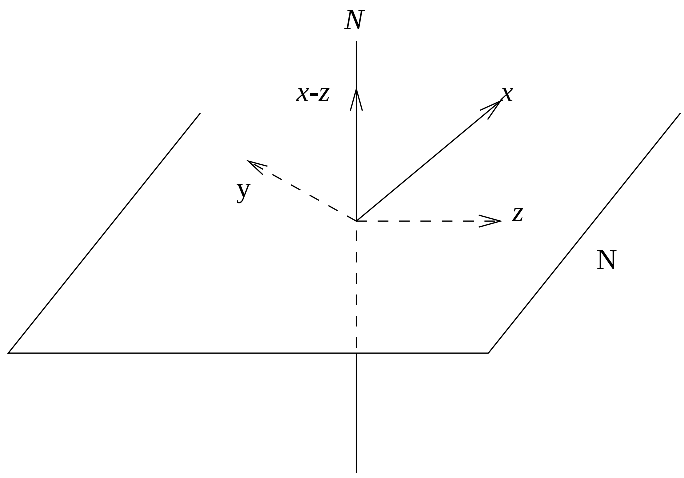

Figure 7.3

i.e.

$$x = \underset{\in N}{z} + \underset{\in N^\perp}{(x - z)}$$

uniquely, as required. ◀

Lemma 7.1. *Let $(\cdot|\cdot)$ be a symmetric scalar product, not identically zero on a vector space M over a field K of characteristic $\neq 2$. (i.e. $1+1 \neq 0$). Then there exists $x \in M$ such that*

$$(x|x) \neq 0.$$

Proof ▶ Choose $x, y \in M$ such that $(x|y) \neq 0$. Then

$$(x + y|x + y) = (x|x) + (x|y) + (y|x) + (y|y).$$

Hence $(x + y|x + y), (x|x), (y|y)$ are not all zero. Hence result. ◀

Theorem 7.2. *Let* $(\cdot|\cdot)$ *be a symmetric scalar product on a finite-dimensional vector space* M*. Then* M *has a basis of mutually orthogonal vectors:*

$$(u_i|u_j) = 0 \quad \text{if } i \neq j,$$

i.e. the scalar product has a diagonal *matrix*

$$\begin{pmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & \alpha_n \end{pmatrix},$$

where $\alpha_i = (u_i|u_i)$*.*

Proof ► Theorem holds if $(x|y) = 0$ for all $x, y \in M$. So suppose $(\cdot|\cdot)$ is not identically zero.

Now we use induction on $\dim M$. Theorem holds if $\dim M = 1$. So assume $\dim M = n > 1$, and that the theorem holds for all spaces of dimension less than n.

Choose $u_1 \in M$ such that

$$(u_1|u_1) = \alpha_1 \neq 0.$$

Let N be the subspace generated by u_1. $(\cdot|\cdot)_N$ has 1×1 matrix (α_1), and therefore is non-singular. Therefore

$$\begin{aligned} M &= N \oplus N^\perp \\ \dim : n &= 1 + n - 1. \end{aligned}$$

By the induction hypothesis $N^\perp$ has basis

$$u_2, \dots, u_n$$

(say) of mutually orthogonal vectors. Therefore $u_1, u_2, \dots, u_n$ is a basis for M of mutually orthogonal vectors, as required. ◄

If M is a complex vector space, we can put

$$w_i = \frac{u_i}{\sqrt{\alpha_i}}$$

for each $\alpha_i > 0$. Then $(w_i|w_i) = 1$ or 0, and rearranging we have a basis wrt which $(\cdot|\cdot)$ has matrix

$$\left(\begin{array}{cc} \boxed{\begin{matrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{matrix}} & 0 \\ 0 & \boxed{\begin{matrix} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{matrix}} \end{array}\right)$$

($r \times r$ diagonal block top left), and the associated quadratic form is a sum of squares:

$$(w^1)^2 + \cdots + (w^r)^2.$$

If M is a real vector space, we can put

$$w_i = \begin{cases} u_i/\sqrt{\alpha_i} & \alpha_i > 0; \\ u_i/\sqrt{-\alpha_i} & \alpha_i < 0; \\ u_i & \alpha_i = 0. \end{cases}$$

Then $(w_i|w_i) = \pm 1$ or 0, and rearranging we have a basis wrt which $(\cdot|\cdot)$ has matrix

$$\left(\begin{array}{ccc} \boxed{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}} & 0 & 0 \\ 0 & \boxed{\begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix}} & 0 \\ 0 & 0 & \boxed{\begin{matrix} 0 & & \\ & \ddots & \\ & & 0 \end{matrix}} \end{array}\right),$$

and the associated quadratic form is a sum and difference of squares:

$$(w^1)^2 + \cdots + (w^r)^2 - (w^{r+1})^2 - \cdots - (w^{r+s})^2.$$

Example: Let $(\cdot|\cdot)$ be a scalar product on a 3-dimensional space M which has matrix

$$A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 0 & -1 \\ 2 & -1 & -3 \end{pmatrix}$$

wrt a basis with coordinate functions x, y, z.

To find new coordinate functions wrt which $(\cdot|\cdot)$ has a diagonal matrix.

Method: Take the associated quadratic form

$$F = 4x^2 - 3z^2 + 4xy + 4xz - 2yz,$$

and write it as a sum and difference of squares, by 'completing squares'. We have:

$$\begin{aligned} F &= 4(x^2 + xy + xz) - 3z^2 - 2yz \\ &= 4(x + \tfrac{1}{2}y + \tfrac{1}{2}z)^2 - y^2 - z^2 - 2yz - 3z^2 - 2yz \\ &= 4(x + \tfrac{1}{2}y + \tfrac{1}{2}z)^2 - (y^2 + 4yz + 4z^2) \\ &= 4(x + \tfrac{1}{2}y + \tfrac{1}{2}z)^2 - (y + 2z)^2 + 0z^2 \\ &= 4u^2 - v^2 + 0w. \end{aligned}$$

Therefore $(\cdot|\cdot)$ has diagonal matrix

$$D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

wrt to coordinate functions

$$\begin{aligned} u &= x + \tfrac{1}{2}y + \tfrac{1}{2}z, \\ v &= y + 2z, \\ w &= z. \end{aligned}$$

The transition matrix is

$$P = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Check: $P^tDP = A$?

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 2 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 0 & -1 \\ 2 & -1 & -3 \end{pmatrix}.$$

For a symmetric scalar product on a real vector space the number of $+$ signs, and the number of $-$ signs, when the matrix is diagonalised, is independent of the coordinates chosen:

Theorem 7.3 (Sylvester's Law of Inertia). *Let $u_1, \ldots, u_n$ and $w_1, \ldots, w_n$ be bases for a real vector space, and let*

$$\begin{aligned} F &= (u^1)^2 + \cdots + (u^r)^2 - (u^{r+1})^2 - \cdots - (u^{r+s})^2 \\ &= (w^1)^2 + \cdots + (w^t)^2 - (w^{t+1})^2 - \cdots - (w^{t+k})^2 \end{aligned}$$

be a quadratic form diagonalised by each of the two bases. Then $r = t$ and $s = k$.

Proof ► Suppose $r \neq t, r > t$ (say). The space of solutions of the $n - r + t$ homogeneous linear equations

$$u^{r+1} = 0, \ldots, u^n = 0, w^1 = 0, \ldots, w^t = 0$$

has dimension at least

$$n - (n - r + t) = r - t > 0.$$

Therefore there exists a non-zero solution x so

$$\begin{aligned} F(x) &= (u^1(x))^2 + \cdots + (u^r(x))^2 > 0 \\ &= -(w^{t+1}(x))^2 - \cdots - (w^{t+k}(x))^2 \leq 0, \end{aligned}$$

which is clearly a contradiction. Therefore $r = t$, and similarly $s = k$. ◄

7.5 Special Spaces

Definition. A real vector space M with a symmetric scalar product $(\cdot|\cdot)$ is called a *Euclidean space* if the associated quadratic form is *positive definite*, i.e.

$$F(x) = (x|x) > 0 \quad \text{for all } x \neq 0,$$

i.e. there exists basis $u_1, \ldots, u_n$ such that $(\cdot|\cdot)$ has matrix

$$\begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & 1 \end{pmatrix}$$

(all $+$ signs).

$$F = (u^1)^2 + \cdots + (u^n)^2,$$
$$(u_i|u_j) = \delta^i_j,$$

i.e. $u_1, \ldots, u_n$ is *orthonormal.*

We write

$$\|x\| = \sqrt{(x|x)} \quad (x \in M),$$

and call it the *norm* of x. We have

$$\|x + y\| \leq \|x\| + \|y\| \quad (\textit{Triangle Inequality}).$$

Thus M is a normed vector space, and therefore a metric space, and therefore a toplogical space.

The scalar product also satisfies:

$$|(x|y)| \leq \|x\|\|y\| \quad (\textit{Schwarz Inequality}).$$

We define the *angle* θ between two non-zero vectors x, y by:

$$\frac{(x|y)}{\|x\|\|y\|} = \cos\theta \quad (0 \leq \theta \leq \pi)$$

(see Figure 7.4).

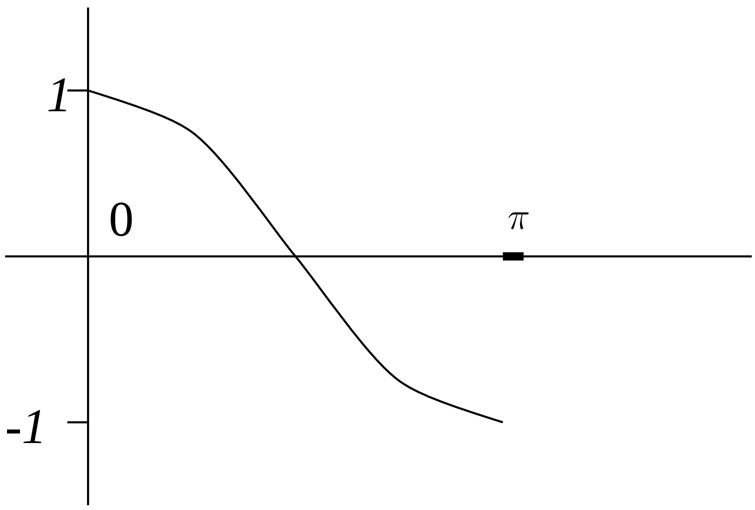

Figure 7.4

If M is an n-dimensional vector space with scalar product having an orthonormal basis (e.g. a complex vector space or a Euclidean vector space) then the transition matrix P from one orthonormal basis to another satisfies:

$$P^t \underset{\text{new}}{I} P = \underset{\text{old}}{I},$$

i.e.

$$P^t P = I$$

i.e. P is an *orthogonal matrix*, i.e.

$$\begin{pmatrix} \cdots i^{th} \text{ col of } P \cdots \\ \\ \end{pmatrix} \begin{pmatrix} \vdots \\ j^{th} \\ \text{col} \\ \text{of } P \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{pmatrix},$$

i.e.

$$(i^{th} \text{ col of } P).(j^{th} \text{ col of } P) = \delta_{ij},$$

i.e. the columns of P form an orthonormal basis of K^n.

Also

$$\begin{aligned} P \text{ orthonormal} &\Leftrightarrow P^t = P^{-1} \\ &\Leftrightarrow PP^t = I \\ &\Leftrightarrow \text{ the rows of } P \text{ form an orthonormal basis of } K^n. \end{aligned}$$

Definition. A real 4-dimensional vector space M with scalar product $(\cdot|\cdot)$ of type $+++-$ is called a *Minkowski space*. A basis u_1, u_2, u_3, u_4 is called a *Lorentz basis* if wrt u_i the scalar product has matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

i.e.

$$F = (u^1)^2 + (u^2)^2 + (u^3)^2 - (u^4)^2.$$

The transition matrix P from one Lorentz basis to another satisfies:

$$P^t \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Such a matrix P is called a *Lorentz matrix*.

Example: On $\mathbb{C}^n$ we define the *hermitian dot product* $(x|y)$ of vectors

$$x = (\alpha_1, \dots, \alpha_n), \quad y = (\beta_1, \dots, \beta_n)$$

to be

$$(x|y) = \alpha_1\overline{\beta_1} + \cdots + \alpha_n\overline{\beta_n}.$$

This has the property of being positive definite, since:

$$(x|x) = \alpha_1\overline{\alpha_1} + \cdots + \alpha_n\overline{\alpha_n} = \|\alpha_1\|^2 + \cdots + \|\alpha_n\|^2 > 0 \quad \text{if } x \neq 0.$$

More generally:

Definition. If M is a complex vector space then a *hermitian scalar product* $(\cdot|\cdot)$ on M is a function

$$M \times M \to \mathbb{C}$$

such that

(i) $(x + y|z) = (x|z) + (y|z)$,

(ii) $(\alpha x|z) = \alpha(x|z)$,

(iii) $(x|y + z) = (x|y) + (x|z)$,

(iv) $(x|\alpha y) = \overline{\alpha}(x|y)$,

(v) $\overline{(x|y)} = (y|x)$.

(i) and (ii) imply linear in the first variable, (iii) and (iv) imply conjugate-linear in the second variable, (v) implies conjugate-symmetric.

If, in addition,

$$(x|x) > 0$$

for all $x \neq 0$ then we call $(\cdot|\cdot)$ a *positive definite* hermitian scalar product.

Definition. A complex vector space M with a positive definite hermitian scalar product $(\cdot|\cdot)$ is called a *Hilbert space*.

Note. For a finite dimensional complex space M with an hermitian form $(\cdot|\cdot)$ we can prove (in exactly the same way as for a real space with symmetric scalar product):

1. There exists basis wrt which $(\cdot|\cdot)$ has matrix

$$\begin{pmatrix} \boxed{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}} & 0 & 0 \\ 0 & \boxed{\begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix}} & 0 \\ 0 & 0 & \boxed{\begin{matrix} 0 & & \\ & \ddots & \\ & & 0 \end{matrix}} \end{pmatrix}.$$

2. The number of $+$ signs and the number of $-$ signs are each uniquely determined by $(\cdot|\cdot)$.

3. M is a Hilbert space iff all the signs are $+$.

Thus M is a Hilbert space iff M has an orthonormal basis. The transition matrix P from one orthonormal basis to another satisfies:

$$P^t \underset{\text{new}}{I} \overline{P} = \underset{\text{old}}{I},$$

i.e.

$$P^t \overline{P} = I.$$

Such a matrix is called a *unitary matrix*.

A Hilbert space M is a normed space, hence a metric space, hence a topological space if we define:

$$\|x\| = \sqrt{(x|x)}.$$

To test how many $+, -$ signs a quadratic form has we can use determinants:

Example:

$$F = ax^2 + 2bxy + cy^2 = a\left(x + \frac{b}{a}y\right)^2 + \frac{ac - b^2}{a}y^2$$

on a 2-dimensional space, with coordinate functions x, y and matrix $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$. Therefore

$$\begin{aligned} ++ \quad &\Leftrightarrow a > 0, \quad \begin{vmatrix} a & b \\ b & c \end{vmatrix} > 0, \\ -- \quad &\Leftrightarrow a < 0, \quad \begin{vmatrix} a & b \\ b & c \end{vmatrix} > 0, \\ +- \quad &\Leftrightarrow \quad \begin{vmatrix} a & b \\ b & c \end{vmatrix} < 0. \end{aligned}$$

More generally:

Theorem 7.4 (Jacobi's Theorem). *Let F be a quadratic form on a real vector space M, with symmetric matrix g_{ij} wrt basis u_i. Suppose each of the determinants*

$$\Delta_i = \begin{vmatrix} g_{11} & \cdots & g_{1i} \\ \vdots & & \vdots \\ g_{i1} & \cdots & g_{ii} \end{vmatrix}$$

is non-zero ($i = 1, \dots, n$). Then there exists a basis w_i such that F has matrix

$$\begin{pmatrix} \frac{1}{\Delta_1} & & & \\ & \frac{\Delta_1}{\Delta_2} & & \\ & & \ddots & \\ & & & \frac{\Delta_{n-1}}{\Delta_n} \end{pmatrix},$$

i.e.

$$F = \frac{1}{\Delta_1}(w^1)^2 + \frac{\Delta_1}{\Delta_2}(w^2)^2 + \cdots + \frac{\Delta_{n-1}}{\Delta_n}(w^n)^2.$$

Thus

$$\begin{aligned} &F \text{ is +ve definite} \Leftrightarrow \Delta_1, \Delta_2, \dots, \Delta_n \text{ all positive}, \\ &F \text{ is −ve definite} \Leftrightarrow \Delta_1 < 0, \Delta_2 > 0, \Delta_3 < 0, \dots. \end{aligned}$$

Proof ▶ $F(x) = (x|x)$, where $(\cdot|\cdot)$ is a symmetric scalar product, $(u_i|u_j) = g_{ij}$. Let

$$N_i = \mathcal{S}(u_1, \dots, u_i).$$

$(\cdot|\cdot)_{N_i}$ is non-singular, since $\Delta_i \neq 0$ for $i = 1, \dots, n$.

Now

$$\{0\} \subset N_1 \subset N_2 \subset \cdots \subset N_{i-1} \subset N_i \subset \cdots \subset N_n = M.$$

Therefore

$$\begin{aligned} N_i &= N_{i-1} \oplus (N_i \cap N_{i-1}^{\perp}) \\ \dim : i &= (i-1) + 1. \end{aligned}$$

Choose non-zero $w_i \in N_i \cap N_{i-1}^{\perp}$. Then

$$w_1, \ldots, w_{i-1}, w_i, \ldots, w_n$$

are mutually orthogonal, and w_i is orthogonal to $u_1, \ldots, u_{i-1}$. Therefore w_i is *not* orthogonal to u_i, since $(\cdot|\cdot)$ is non-singular. Therefore we can choose w_i such that $(u_i|w_i) = 1$.

It remains to show that

$$(w_i|w_i) = \frac{\Delta_{i-1}}{\Delta_i}.$$

To do this we write

$$\lambda_1 u_1 + \cdots + \lambda_{i-1} u_{i-1} + \lambda_i u_i = w_i.$$

Taking scalar product with $w_i, u_1, u_2, \ldots, u_i$ we get:

$$\begin{aligned} 0 + \cdots + 0 + \lambda_i &= (w_i|w_i) \\ \lambda_1 g_{11} + \cdots + \lambda_{i-1} g_{1,i-1} + \lambda_i g_{1i} &= 0 \\ \lambda_1 g_{21} + \cdots + \lambda_{i-1} g_{2,i-1} + \lambda_i g_{2i} &= 0 \\ &\vdots \\ \lambda_1 g_{i-1,1} + \cdots + \lambda_{i-1} g_{i-1,i-1} + \lambda_i g_{i-1,i} &= 0 \\ \lambda_1 g_{i1} + \cdots + \lambda_{i-1} g_{i,i-1} + \lambda_i g_{ii} &= 1 \end{aligned}$$

Therefore

$$(w_i|w_i) = \lambda_i = \frac{\begin{vmatrix} g_{11} & \cdots & g_{1,i-1} & 0 \\ \vdots & & \vdots & \vdots \\ g_{i-1,1} & \cdots & g_{i-1,i-1} & 0 \\ g_{i1} & \cdots & g_{i,i-1} & 1 \end{vmatrix}}{\begin{vmatrix} g_{11} & \cdots & g_{1,i-1} & g_{1i} \\ \vdots & & \vdots & \vdots \\ g_{i-1,1} & \cdots & g_{i-1,i-1} & g_{i-1,i} \\ g_{i1} & \cdots & g_{i,i-1} & g_{ii} \end{vmatrix}} = \frac{\Delta_{i-1}}{\Delta_i},$$

as requried. ◄

This has an application in Calculus:

Theorem 7.5 (Criteria for local maxima or minima). *Let f be a scalar field on a manifold X such that $df_X = 0$, and let y^i be coordinates on X at a. Put*

$$\Delta_i = \begin{vmatrix} \frac{\partial^2 f}{\partial {y^1}^2} & \cdots & \frac{\partial^2 f}{\partial y^1 \partial y^i} \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial y^i \partial y^1} & \cdots & \frac{\partial^2}{\partial {y^i}^2} \end{vmatrix}.$$

Then

1. *If $\Delta_i(a) > 0$ for $i = 1, \ldots, n$ then there exists open nbd V of a such that*
$$f(x) > f(a) \quad \text{for all } x \in V,\ x \neq a,$$
i.e. a is a local minima *of f;*

2. *If $\Delta_1(a) < 0, \Delta_2(a) > 0, \Delta_3(a) < 0, \ldots$ then there exists open nbd V of a such that*
$$f(x) < f(a) \quad \text{for all } x \in V,\ x \neq a,$$
i.e. a is a local maxima *of f.*

To make sure that $\|x\| = \sqrt{(x|x)}$ is a norm on a Euclidean or a Hilbert space we need to show that the triangle inequality holds.

Theorem 7.6. *Let M be a Euclidean or a Hilbert space. Then*

(i) $|(x|y)| \leq \|x\|\|y\|$ *Schwarz,*

(ii) $\|x + y\| \leq \|x\| + \|y\|$ *Triangle.*

Proof ▶

(i) Let $x, y \in M$. Then
$$(x|y) = |(x|y)|e^{i\theta}, \quad (y|x) = |(x|y)|e^{-i\theta}$$
(say). So for all $\lambda \in \mathbb{R}$ we have:
$$\begin{aligned} 0 &\leq (\lambda e^{-i\theta}x + y|\lambda e^{-i\theta}x + y) \\ &= \|x\|^2\lambda^2 + \lambda e^{-i\theta}(x|y) + \lambda e^{i\theta}(y|x) + \|y\|^2 \\ &= \|x\|^2 + 2\lambda|(x|y)| + \|y\|^2. \end{aligned}$$
Therefore
$$|(x|y)|^2 \leq \|x\|^2\|y\|^2 \quad (\text{`}b^2 \leq 4ac\text{'}).$$
Therefore
$$|(x|y)| \leq \|x\|\|y\|.$$

(ii)

$$\begin{aligned}\|x+y\|^2 &= (x+y|x+y)\\ &= \|x\|^2 + (x|y) + (y|x) + \|y\|^2\\ &\le \|x\|^2 + 2\|x\|\|y\| + \|y\|^2\\ &= (\|x\| + \|y\|)^2.\end{aligned}$$

Therefore

$$\|x+y\| \le \|x\| + \|y\|.$$

◀

Chapter 8

Linear Operators 2

8.1 Adjoints and Isometries

Let M be a finite dimensional vector space with a fixed non-singular symmetric or hermitian scalar product $(\cdot|\cdot)$. Recall that if

$$M \xrightarrow{T} M$$

is a linear operator then the *adjoint* of T is the operator

$$M \xrightarrow{T^*} M,$$

which satisfies

$$(x|Ty) = (T^*x|y)$$

for all $x, y \in M$.

If $(\cdot|\cdot)$ has matrix G wrt basis u_i and T has matrix A then T^* has matrix

$$A^* = G^{-1}A^tG \quad (\overline{G^{-1}A^tG} \text{ in hermitian case})$$

because

$$X^tGAY = X^tGAG^{-1}GY = [G^{-1}A^tGX]^tGY,$$

and similarly

$$X^tG\overline{AY} = X^tG\overline{A}G^{-1}G\overline{Y} = [\overline{G^{-1}A^tG}X]^tG\overline{Y}.$$

Examples:

1. M Euclidean, basis orthonormal:

$$A^* = A^t.$$

2. M Hilbert, basis orthonormal:

$$A^* = \overline{A}^t.$$

3. M Minkowski, basis Lorentz:

$$A^* = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} A^t \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Definition. An operator $M \xrightarrow{T} M$ is called an *isometry* if

$$(Tx|Ty) = (x|y) \quad \text{for all } x, y \in M,$$

i.e. T preserves $(\cdot|\cdot)$, i.e.

$$(T^*Tx|y) = (x|y),$$

i.e.

$$T^*T = 1,$$

i.e.

$$T^* = T^{-1}.$$

Examples:

1. M Euclidean, basis orthonormal, A matrix of T:

$$T \text{ is an isometry} \Leftrightarrow A^tA = I,$$

   i.e. A is an orthogonal matrix.

2. M Hilbert, basis orthonormal, A matrix of T:

$$T \text{ is an isometry} \Leftrightarrow \overline{A}^tA = I,$$

   i.e. A is a unitary matrix.

3. M Minkowski, basis Lorentz, A matrix of T:
$$T \text{ is an isometry} \Leftrightarrow GA^tGA = I \Leftrightarrow A^tGA = G,$$
i.e. A is a Lorentz matrix.

Definition. An isometry of a Euclidean space is called an *orthogonal transformation.* An isometry of a Hilbert space is called a *unitary transformation.* An isometry of a Minkowski space is called a *Lorentz transformation.*

Definition. An operator $M \xrightarrow{T} M$ is called *self-adjoint* if
$$T^* = T,$$
i.e.
$$(Tx|y) = (x|Ty) \quad \text{for all } x, y \in M,$$
i.e.
$$(u_i|Tu_j) = (Tu_i|u_j) = (u_j|Tu_i),$$
i.e. covariant components of T are symmetric.

(In quantum mechanics physical quantities are always represented by self-adjoint operators).

Examples:

1. M Euclidean, basis orthonormal, A matrix of T:
$$T \text{ is self-adjoint} \Leftrightarrow A^t = A,$$
i.e. A is symmetric.

2. M Hilbert, basis orthonormal, A matrix of T:
$$T \text{ is self-adjoint} \Leftrightarrow \overline{A}^t = A,$$
i.e. A is hermitian.

3. M Minkowski, basis Lorentz, A matrix of T:
$$T \text{ is self-adjoint} \Leftrightarrow GA^tG = A.$$

Summary: Let $M \xrightarrow{T} M$ have matrix A wrt orthonormal or Lorentz basis. Then:

Space:	Euclidean	Hilbert	Minkowski
Matrix of T^*:	A^t	$\overline{A}^t$	GA^tG
T self-adjoint:	$A^t = A$ A symmetric	$\overline{A}^t = A$ A Hermitian	$GA^tG = A$
T an isometry:	$A^tA = I$ A orthogonal	$\overline{A}^tA = I$ A unitary	$A^tGA = G$ A Lorentz

8.2 Eigenvalues and Eigenvectors

Definition. A vector space $N \subset M$ is called *invariant* under a linear operator $M \xrightarrow{T} M$ if

$$T(N) \subset N,$$

i.e. $x \in N \Rightarrow Tx \in N$.

A non-zero vector in a 1-dimensional invariant subspace under T is called an eigenvector of T:

(i) $x \in M$ is called an *eigenvector of* T*, with eigenvalue* λ if

(a) $x \neq 0$,

(b) $Tx = \lambda x$, where λ is a scalar (see Figure 8.1);

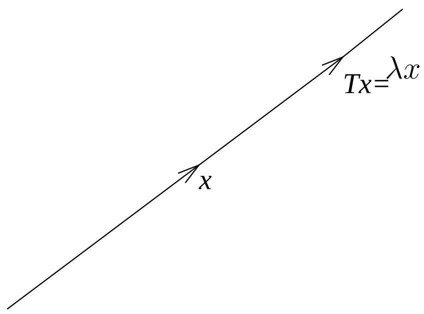

Figure 8.1

(ii) $\lambda \in K$ is called an *eigenvalue of* T if there exists $x \neq 0$ such that

$$Tx = \lambda x,$$

i.e.

$$(T - \lambda 1)x = 0,$$

i.e.

$$\ker(T - \lambda 1) \neq \{0\}.$$

$$\ker(T - \lambda 1) = \{x \in M : Tx = \lambda x\}$$

is called the λ-*eigenspace of* T. It is the vector subspace consisting of all eigenvectors of T having eigenvalue λ, together with the zero vector.

Definition. If $M \xrightarrow{T} M$ is a linear operator on a vector space of finite dimension n, with matrix $A = \left(\alpha^i_j\right)$ wrt basis u_i, then the polynomial of degree n with coefficients in K:

$$\operatorname{char} T = \det \begin{vmatrix} \alpha^1_1 - X & \alpha^1_2 & \dots & \alpha^1_n \\ \alpha^2_1 & \alpha^2_2 - X & & \vdots \\ \vdots & & \ddots & \vdots \\ \alpha^n_1 & \dots & \dots & \alpha^n_n - X \end{vmatrix} = \det(A - XI)$$

is called the *characteristic polynomial* of T.

$\operatorname{char} T$ is well-defined, independent of choice of basis u_i, since if B is the matrix of T wrt another basis then

$$B = PAP^{-1}.$$

Therefore

$$\begin{aligned} \det(B - XI) &= \det(PAP^{-1} - XI) \\ &= \det P(A - XI)P^{-1} \\ &= \det P \det(A - XI) \det P^{-1} \\ &= \det(A - XI), \end{aligned}$$

since $\det P \det P^{-1} = \det PP^{-1} = \det I = 1$.

Theorem 8.1. *If $M \xrightarrow{T} M$ is a linear operator and $\dim M < \infty$ and $\lambda \in K$ then*

$$\lambda \textit{ is an eigenvalue of } T \Leftrightarrow \lambda \textit{ is a zero of } \operatorname{char} T.$$

Proof ▶ Let T have matrix A wrt basis u_i. Then

$$\begin{aligned} \lambda \text{ is an eigenvalue of } T &\Leftrightarrow \text{ there exists } y \in M \text{ such that } (T - \lambda 1)y = 0 \\ &\Leftrightarrow \text{ there exists } Y \in K^n \text{ such that } (A - \lambda I)Y = 0 \\ &\Leftrightarrow \det(A - \lambda I) = 0 \\ &\Leftrightarrow \lambda \text{ is a zero of } \det(A - XI). \end{aligned}$$

◀

Corollary 8.1. *If T is a linear operator on a finite dimensional complex space then T has an eigenvalue, and therefore eigenvectors.*

Theorem 8.2. *Let $M \xrightarrow{T} M$ be a linear operator on a finite dimensional vector space M. Then T has a diagonal matrix*

$$\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

wrt a basis $u_1, \ldots, u_n$ iff u_i is an eigenvector of T, with eigenvalue λ_i, for $i = 1, \ldots, n$.

Proof ►

$$\begin{aligned} Tu_1 &= \lambda_1 u_1 + 0u_2 + \cdots + 0u_n \\ Tu_2 &= 0u_1 + \lambda_2 u_2 + \cdots + 0u_n \\ &\vdots \\ Tu_n &= 0u_1 + 0u_2 + \cdots + \lambda_n u_n, \end{aligned}$$

hence result. ◄

Theorem 8.3. *Let $M \xrightarrow{T} M$ ba a self-adjoint operator on a Hilbert space M. Then all the eigenvalues of T are real.*

Proof ► Let $Tx = \lambda x,\ x \neq 0,\ \lambda \in \mathbb{C}$. Then

$$\lambda(x|x) = (\lambda x|x) = (Tx|x) = (x|Tx) = (x|\lambda x) = \overline{\lambda}(x|x).$$

$(x|x) \neq 0$. Therefore $\lambda = \overline{\lambda}$. Therefore λ is real. ◄

Corollary 8.2. *Let A ba a hermitian matrix. Then $\mathbb{C}^n \xrightarrow{A} \mathbb{C}^n$ is a self-adjoint operator wrt hermitian dot product. Therefore all the roots of the equation*

$$\det(A - XI) = 0$$

are real.

Corollary 8.3. *Let T be a self-adjoint operator on a finite dimensional Euclidean space. Then T has an eigenvector.*

Proof ► Wrt an orthonormal basis T has a real symmetric matrix A:

$$\overline{A}^t = A^t = A.$$

Therefore A is hermitian. Therefore $\det(A - XI) = 0$ has real roots. Therefore T has an eigenvalue. Therefore T has eigenvectors. ◄

Theorem 8.4. *Let N be invariant under a linear operator $M \xrightarrow{T} M$. Then $N^\perp$ is invariant under T^*.*

Proof ▶ Let $x \in N^\perp$. Then for all $y \in N$ we have:

$$(T^*x|y) = (x|Ty) = 0.$$

Therefore $T^*x \in N^\perp$. ◀

Definition. $M \xrightarrow{T} M$ is a *normal operator* if

$$T^*T = TT^*,$$

i.e. T commutes with T^*.

Examples:

(i) T self-adjoint $\Rightarrow T$ normal.

(ii) T an isometry $\Rightarrow T$ normal.

Theorem 8.5. *Let S, T be commuting linear operators $M \to M$ ($ST = TS$). Then each eigenspace of S is invariant under T.*

Proof ▶

$$Sx = \lambda x \Rightarrow S(Tx) = T(Sx) = T(\lambda x) = \lambda(Tx),$$

i.e. $x \in \lambda$-eigenspace of $S \Rightarrow Tx \in \lambda$-eigenspace of S. ◀

8.3 Spectral Theorem and Applications

Theorem 8.6 (Spectral theorem). *Let $M \xrightarrow{T} M$ be* either *a self-adjoint operator on a finite dimensional Euclidean space* or *a normal operator on a finite dimensional Hilbert space. Then M has an orthonormal basis of eigenvectors of T.*

Proof ▶ (By induction on $\dim M$). True for $\dim M = 1$. Let $\dim M = n$, and assume the theorem holds for spaces of dimension $\leq n - 1$.

Let λ be an eigenvalue of T, M_λ the λ-eigenspace. $(\cdot|\cdot)_{M_\lambda}$ is non-singular, since $(\cdot|\cdot)$ is positive definite. Therefore

$$M = M_\lambda \oplus M_\lambda^\perp.$$

M_λ is T-invariant. Therefore $M_\lambda^\perp$ is T^*-invariant. T^* commutes with T. Therefore M_λ is T^*-invariant. Therefore $M_\lambda^\perp$ is T-invariant.

Now

$$(T^*x|y) = (x|Ty)$$

for all $x, y \in M_\lambda^\perp$. Therefore $(T^*)_{M_\lambda^\perp}$ is the adjoint of $T_{M_\lambda^\perp}$. Therefore

$$T \text{ self-adjoint} \Rightarrow T_{M_\lambda^\perp} \text{ is self-adjoint,}$$

and

$$T \text{ normal} \Rightarrow T_{M_\lambda^\perp} \text{ is normal.}$$

But $\dim M_\lambda^\perp \leq n - 1$. Therefore, by induction hypothesis $M_\lambda^\perp$ has an orthonormal basis of eigenvectors of T. Therefore $M = M_\lambda \oplus M_\lambda^\perp$ has an orthonormal basis of eigenvectors of T. ◄

Applications:

1. Let A be a real symmetric $n \times n$ matrix. Then

   (i) $\mathbb{R}^n$ has an orthonormal basis of eigenvectors $u_1, \ldots, u_n$ of A, with eigenvalues $\lambda_1, \ldots, \lambda_n$ (say),

   (ii) if P is the matrix having $u_1, \ldots, u_n$ as rows then P is an orthogonal matrix and

   $$PAP^{-1} = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & \lambda_n \end{pmatrix} = Q^t A Q,$$

   where $Q = P^{-1}$.

   Proof ► $\mathbb{R}^n$ is a Euclidean space wrt the dot product, $e_1, \ldots, e_n$ is an orthonormal basis. Operator $\mathbb{R} \xrightarrow{A} \mathbb{R}^n$ has symmetric matrix A wrt orthonormal basis $e_1, \ldots, e_n$. Therefore A is self-adjoint. Therefore $\mathbb{R}^n$ has an orthonormal basis $u_1, \ldots, u_n$, with eigenvalues $\lambda_1, \ldots, \lambda_n$.

   Let P be the transition matrix from orthonormal $e_1, \ldots, e_n$ to orthonormal $u_1, \ldots, u_n$, with inverse matrix Q. P is an orthogonal matrix, and therefore

   $$Q = P^{-1} = P^t.$$

Q is the transition matrix from u_i to e_i. Therefore

$$\begin{aligned} u_j = q_j^1 e_1 + \cdots + q_j^n e_n &= (q_j^1, \ldots, q_j^n) \\ &= j^{th} \text{ column of } Q \\ &= j^{th} \text{ row of } P. \end{aligned}$$

Matrix of operator A wrt basis u_i is:

$$PAP^{-1} = PAP^t = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & \lambda_n \end{pmatrix}.$$

◄

2. (**Principal axes theorem**) *Let F be a quadratic form on a finite dimensional Euclidean space M. Then M has an orthonormal basis $u_1, \ldots, u_n$ which diagonalises F:*

$$F = \lambda_1 (u^1)^2 + \cdots + \lambda_n (u^n)^2.$$

Such a basis is called a set of principal axes *for F.*

Proof ► $F(x) = B(x, x)$, where B is a symmetric bilinear form. Raising an index of B gives a self-adjoint operator T:

$$(x|Ty) = B(x, y) = (Tx|y).$$

Let $u_1, \ldots, u_n$ be an orthonormal basis of M of eigenvectors of T, with eigenvalues $\lambda_1, \ldots, \lambda_n$ (say). Then wrt u_i the quadratic form F has matrix:

$$B(u_i, u_j) = (u_i|Tu_j) = (u_i|\lambda_j u_j) = \lambda_j \delta_j^i,$$

i.e.

$$\begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & \lambda_n \end{pmatrix},$$

as required. ◄

Note. If F has matrix $A = (\alpha_{ij})$, and $(\cdot|\cdot)$ has matrix $G = (g_{ij})$ then T has matrix

$$({\alpha^i}_j) = (g^{ik} \alpha_{kj}) = G^{-1} A.$$

Therefore $\lambda_1, \ldots, \lambda_n$ are the roots of

$$\det(G^{-1}A - XI) = 0,$$

i.e.

$$\det(A - XG) = 0.$$

3. Consider the surface

$$ax^2 + by^2 + cz^2 + 2hxy + 2gxz + 2fyz = k \quad (k > 0)$$

in $\mathbb{R}^3$. The LHS is a quadratic form with matrix

$$A = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$$

wrt usual coordinate functions x, y, z. By the principal axes theorem we can choose new orthonormal coordinates X, Y, Z such that equation becomes:

$$\lambda_1 X^2 + \lambda_2 Y^2 + \lambda_3 Z^2 = k,$$

where $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of A.

The surface is:

an *ellipsoid* if $\lambda_1, \lambda_2, \lambda_3$ are all > 0, i.e. if the quadratic form is positive definite, i.e.

$$a > 0,\ ab - h^2 > 0,\ \det A > 0 \quad \text{by Jacobi;}$$

a *hyperboloid of 1-sheet* (see Figure 8.2) if the quadratic form is of type $++-$ (e.g. $X^2 + Y^2 = Z^2 + 1$), i.e.

$$\begin{aligned} & a > 0,\ ab - h^2 > 0,\ \det A < 0 \\ \text{or}\ \ & a > 0,\ ab - h^2 < 0,\ \det A < 0 \\ \text{or}\ \ & a < 0,\ ab - h^2 < 0,\ \det A < 0; \end{aligned}$$

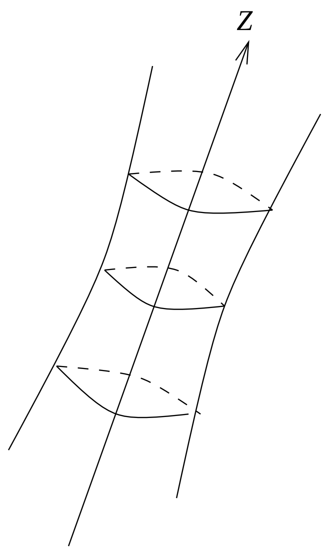

Figure 8.2

a *hyperboloid of 2-sheets* (see Figure 8.3) if the quadratic form is of type $+--$ (e.g. $X^2 + Y^2 = Z^2 - 1$), i.e.

$$\begin{aligned} & a > 0,\ ab - h^2 < 0,\ \det A > 0 \\ \text{or}\quad & a < 0,\ ab - h^2 < 0,\ \det A > 0 \\ \text{or}\quad & a < 0,\ ab - h^2 > 0,\ \det A > 0. \end{aligned}$$

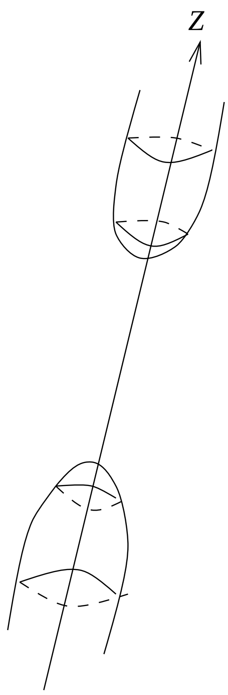

Figure 8.3

Chapter 9

Skew-Symmetric Tensors and Wedge Product

9.1 Skew-Symmetric Tensors

Definition. A bijective map

$$\sigma : \{1, 2, \ldots, r\} \to \{1, 2, \ldots, r\}$$

is called a *permutation of degree* r. The group $\mathcal{S}_r$ of all permutations of degree r is called the *symmetric group* of degree r. Thus $\mathcal{S}_r$ is a group of order $r!$.

Let $\mathcal{T}^r M$ denote the space of all tensors over M of type

$$M \times M \times \cdots \times M \to K.$$

Thus $\mathcal{T}^r M$ consists of all tensors T with components

$$T(u_{i_1}, \ldots, u_{i_r}) = \alpha_{i_1 \ldots i_r} \quad (r \text{ lower indices}),$$
$$T = \alpha_{i_1 \ldots i_r} u^{i_1} \otimes \cdots \otimes u^{i_r}.$$

$u^{i_1} \otimes \cdots \otimes u^{i_r}$ is a basis for $\mathcal{T}^r M$.

For each $\sigma \in \mathcal{S}_r$, and each $T \in \mathcal{T}^r M$ we define $\sigma.T \in \mathcal{T}^r M$ by:

$$(\sigma.T)(x_1, \ldots, x_r) = T(x_{\sigma(1)}, \ldots, x_{\sigma(r)}).$$

If T has components $\alpha_{i_1 \ldots i_r}$ then $\sigma.T$ has components $\beta_{i_1 \ldots i_r}$, where

$$\beta_{i_1 \ldots i_r} = (\sigma.T)(u_{i_1}, \ldots, u_{i_r}) = T(u_{i_{\sigma(1)}}, \ldots, u_{i_{\sigma(r)}}) = \alpha_{i_{\sigma(1)}} \ldots \alpha_{i_{\sigma(r)}}.$$

Theorem 9.1. *The group $\mathcal{S}_r$ acts on $\mathcal{T}^r M$ by linear transformations, i.e.*

(i) $\sigma.(\alpha T + \beta S) = \alpha(\sigma.T) + \beta(\sigma.S)$,

(ii) $\sigma.(\tau.T) = (\sigma\tau).T$,

(iii) $1.T = T$

for all $\alpha, \beta \in K$, $\sigma, \tau \in \mathcal{S}_r$, $S, T \in \mathcal{T}^r M$.

Proof ▶ e.g. (ii)

$$\begin{aligned}[\sigma.(\tau.T)](x_1, \ldots, x_r) &= (\tau.T)[x_{\sigma(1)}, \ldots, x_{\sigma(r)}] \\ &= T(x_{\sigma(\tau(1))}, \ldots, x_{\sigma(\tau(r))}) \\ &= [(\sigma\tau).T](x_1, \ldots, x_r).\end{aligned}$$

Therefore $\sigma.(\tau.T) = (\sigma\tau).T$. ◀

Note. If $\sigma \in \mathcal{S}_r$, we put

$$\epsilon^\sigma = \left\{ \begin{array}{ll} +1 & \text{if } \sigma \text{ is an even permutation;} \\ -1 & \text{if } \sigma \text{ is an odd permutation} \end{array} \right\} = \textit{ sign} \text{ of } \sigma.$$

We have:

(i) $\epsilon^{\sigma\tau} = \epsilon^\sigma \epsilon^\tau$,

(ii) $\epsilon^1 = 1$,

(iii) $\epsilon^{\sigma^{-1}} = \epsilon^\sigma$.

Definition. $T \in \mathcal{T}^r M$ is *skew-symmetric* if

$$\sigma.T = \epsilon^\sigma T \quad \text{for all } \sigma \in \mathcal{S}_r,$$

i.e.

$$T(x_{\sigma(1)}, \ldots, x_{\sigma(r)}) = \epsilon^\sigma T(x_1, \ldots, x_r)$$

for all $\sigma \in \mathcal{S}_r$, $x_1, \ldots, x_r \in M$, i.e. the components $\alpha_{i_1 \ldots i_r}$ of T satisfy:

$$\alpha_{i_{\sigma(1)} \ldots i_{\sigma(r)}} = \epsilon^\sigma \alpha_{i_1 \ldots i_r}.$$

Example: $T \in \mathcal{T}^3 M$, with components α_{ijk} is skew-symmetric iff

$$\alpha_{ijk} = -\alpha_{jik} = \alpha_{jki} = -\alpha_{kji} = \alpha_{kij} = -\alpha_{ikj}.$$

It follows that if T is skew-symmetric, with components $\alpha_{i_1 \ldots i_r}$ (from now on assume K has characteristic zero, i.e. $\alpha \neq 0 \Rightarrow \alpha + \alpha + \cdots + \alpha \neq 0$) then

1. $\alpha_{i_1 \dots i_r} = 0$ if $i_1, \dots, i_r$ are not all distinct;

2. if we know $\alpha_{i_1 \dots i_r}$ for all *increasing sequences* $i_1 < \cdots < i_r$ then we know $\alpha_{i_1 \dots i_r}$ for *all* sequences $i_1, \dots, i_r$;

3. if T is skew-symmetric, with components $\alpha_{i_1 \dots i_r}$ and S is skew-symmetric, with components $\beta_{i_1 \dots i_r}$, and if $\alpha_{i_1 \dots i_r} = \beta_{i_1 \dots i_r}$ for all *increasing* sequences $i_1 < \cdots < i_r$ then $T = S$.

Theorem 9.2. *Let $T \in \mathcal{T}^r M$. Then*

$$\sum_{\sigma \in \mathcal{S}_r} \epsilon^\sigma \sigma.T$$

is skew-symmetric.

Proof ▶ Let $\tau \in \mathcal{S}_r$. Then

$$\tau.\left(\sum_{\sigma \in \mathcal{S}_r} \epsilon^\sigma \sigma.T \right) = \epsilon^\tau \sum_{\sigma \in \mathcal{S}_r} \epsilon^{\tau\sigma} (\tau\sigma).T = \epsilon^\tau \sum_{\sigma \in \mathcal{S}_r} \sigma \epsilon^\sigma (\sigma.T),$$

as required. ◀

Definition. The linear operator

$$\mathcal{A} : \mathcal{T}^r M \to \mathcal{T}^r M$$

defined by

$$\mathcal{A}T = \frac{1}{r!} \sum_{\sigma \in \mathcal{S}_r} \epsilon^\sigma \sigma.T$$

is called the *skew-symmetriser.*

Example: Let $T \in \mathcal{T}^3 M$ have components α_{ijk}. Then

$$\mathcal{A}T(x, y, z) =$$
$$\frac{1}{6}[T(x, y, z) - T(y, x, z) + T(y, z, x) - T(x, z, y) + T(z, x, y) - T(z, y, x)],$$

and $\mathcal{A}T$ has components

$$\beta_{ijk} = \frac{1}{6}(\alpha_{ijk} - \alpha_{ikj} + \alpha_{jki} - \alpha_{jik} + \alpha_{kij} - \alpha_{kji}).$$

Theorem 9.3. *Let $S \in \mathcal{T}^s M,\ T \in \mathcal{T}^t M$. Then*

(i) $\mathcal{A}[(\mathcal{A}S) \otimes T] = \mathcal{A}[S \otimes T] = \mathcal{A}[S \otimes \mathcal{A}T]$,

(ii) $\mathcal{A}(S \otimes T) = (-1)^{st} \mathcal{A}(T \otimes S)$.

Proof ▶

(i) We first note that if $\tau \in \mathcal{S}_s$ then

$$\begin{aligned}
[(\tau.S) \otimes T](x_1, \ldots, x_s, x_{s+1}, \ldots, x_{s+t}) &= (\tau.S)(x_1, \ldots, x_s) T(x_{s+1}, \ldots, x_{s+t}) \\
&= S(x_{\tau(1)}, \ldots, x_{\tau(s)}) T(x_{s+1}, \ldots, x_{s+t}) \\
&= S(x_{\tau'(1)}, \ldots, x_{\tau'(s)}) T(x_{\tau'(s+1)}, \ldots x_{\tau'(s+t)}) \\
&= [\tau'.(S \otimes T)](x_1, \ldots, x_s, x_{s+1}, \ldots, x_{s+t}),
\end{aligned}$$

where

$$\tau' = \begin{pmatrix} 1 & \ldots & s & s+1 & \ldots & s+t \\ \tau(1) & \ldots & \tau(s) & s+1 & \ldots & s+t \end{pmatrix}.$$

Thus

$$(\tau.S) \otimes T = \tau'.(S \otimes T)$$

and $\epsilon^{\tau'} = \epsilon^{\tau}$.

Now

$$\begin{aligned}
\mathcal{A}[(\mathcal{A}S) \otimes T] &= \frac{1}{(s+t)!} \sum_{\sigma \in \mathcal{S}_{s+t}} \epsilon^{\sigma} \sigma. \left[\left(\frac{1}{s!} \sum_{\tau \in \mathcal{S}_s} \epsilon^{\tau} \tau.S \right) \otimes T \right] \\
&= \frac{1}{s!} \sum_{\tau \in \mathcal{S}_s} \frac{1}{(s+t)!} \sum_{\sigma \in \mathcal{S}_{s+t}} \epsilon^{\sigma\tau'} (\sigma\tau').(S \otimes T) \\
&= \frac{1}{s!} \sum_{\tau \in \mathcal{S}_s} \mathcal{A}(S \otimes T) \\
&= \mathcal{A}(S \otimes T).
\end{aligned}$$

(ii) Let

$$\tau = \begin{pmatrix} 1 & \ldots & s & s+1 & \ldots & s+t \\ t+1 & \ldots & t+s & 1 & \ldots & t \end{pmatrix}$$

so that $\epsilon^{\tau} = (-1)^{st}$. Then

$$\begin{aligned}
[\tau.(S \otimes T)](x_1, \ldots, x_s, x_{s+1}, \ldots, x_{s+t}) &= S \otimes T[x_{t+1}, \ldots, x_{t+s}, x_1, \ldots, x_t] \\
&= T(x_1, \ldots, x_t) S(x_{t+1}, \ldots x_{t+s}) \\
&= (T \otimes S)[x_1, \ldots, x_{t+s}].
\end{aligned}$$

Therefore

$$\tau.(S \otimes T) = T \otimes S.$$

Therefore

$$\begin{aligned}\mathcal{A}(S \otimes T) &= \frac{1}{(s+t)!} \sum_{\sigma \in \mathcal{S}_{s+t}} \epsilon^{\sigma\tau} \sigma\tau.(T \otimes S) \\ &= \epsilon^{\tau} \frac{1}{(s+t)!} \sum_{\sigma \in \mathcal{S}_{s+t}} \epsilon^{\sigma} \sigma.(T \otimes S) \\ &= (-1)^{st} \mathcal{A}(T \otimes S),\end{aligned}$$

as required. ◀

9.2 Wedge Product

Definition. If $S \in \mathcal{T}^s M$ and $T \in \mathcal{T}^t M$, we define their *wedge product* (also called *exterior product*) by

$$S \wedge T = \frac{1}{s!t!} \sum_{\sigma \in \mathcal{S}_{s+t}} \epsilon^{\sigma} \sigma(S \otimes T) = \frac{(s+t)!}{s!t!} \mathcal{A}(S \otimes T).$$

Example: Let $S, T \in M^*$ have components α_i, β_i wrt u_i. Then

$$S \wedge T = S \otimes T - T \otimes S.$$

Therefore

$$S \wedge T[x, y] = S(x)T(y) - T(x)S(y),$$

and $S \wedge T$ has components

$$\gamma_{ij} = S \wedge T[u_i, u_j] = S(u_i)T(u_j) - T(u_i)S(u_j) = \alpha_i\beta_j - \beta_i\alpha_j.$$

Theorem 9.4. *The wedge product has the following properties:*

1. $(R + S) \wedge T = R \wedge T + S \wedge T,$
2. $R \wedge (S + T) = R \wedge S + R \wedge T,$
3. $(\lambda R) \wedge S = \lambda(R \wedge S) = R \wedge (\lambda S),$
4. $R \wedge (S \wedge T) = (R \wedge S) \wedge T,$

5. $S \wedge T = (-1)^{st} T \wedge S$*,*

6. $R_1 \wedge \cdots \wedge R_k = \frac{(r_1+\cdots+r_k)!}{r_1!\ldots r_k!} \mathcal{A}(R_1 \otimes \cdots \otimes R_k)$*.*

$(i), (ii)$ *and* (iii) *imply* bilinear*;* (iv) *implies* associative*;* (v) *implies* graded commutative*.*

Proof ▶ e.g. 4.

$$\begin{aligned}(R \wedge S) \wedge T &= \frac{(r+s+t)!}{(r+s)!t!} \mathcal{A}\left[\frac{(r+s)!}{r!s!}(\mathcal{A}(R \otimes S) \otimes T\right] \\ &= \frac{(r+s+t)!}{r!s!t!} \mathcal{A}(R \otimes S \otimes T) \\ &\overset{\text{sim.}}{=} R \wedge (S \wedge T).\end{aligned}$$

5.

$$S \wedge T = \frac{(s+t)!}{s!t!} \mathcal{A}(S \otimes T) = (-1)^{st} \frac{(t+s)!}{t!s!} \mathcal{A}(T \otimes S) = (-1)^{st} T \wedge S.$$

6. By induction on k: true for $k = 1$, assume true for $k-1$. Then:

$$\begin{aligned}&(R_1 \wedge \cdots \wedge R_{k-1}) \wedge R_k \\ &= \frac{(r_1+\cdots+r_{k-1}+r_k)!}{(r_1+\cdots+r_{k-1})!r_k!} \mathcal{A}\left[\frac{r_1+\cdots+r_{k-1})!}{r_1!\ldots r_{k-1}!} \mathcal{A}((R_1 \otimes \cdots \otimes R_{k-1}) \otimes R_k)\right] \\ &= \frac{(r_1+\cdots+r_k)!}{r_1!\ldots r_k!} \mathcal{A}[R_1 \otimes \cdots \otimes R_k].\end{aligned}$$

◀

Note. For each integer $r > 0$ we write

$M^{(r)}$ for the space of all skew-symmetric tensors of type

$$M \underset{\leftarrow r \rightarrow}{\times \cdots \times} M \to K;$$

$M_{(r)}$ for the space of all skew-symmetric tensors of type

$$M^* \underset{\leftarrow r \rightarrow}{\times \cdots \times} M^* \to K;$$

$M^{(0)} = K = M_{(0)}$.

If $S \in M^{(s)}$ or $S \in M_{(s)}$, we say that S has *degree* s, and we have

$$S \wedge T = (-1)^{st} T \wedge S \quad \text{if } s = \deg S,\ t = \deg T.$$

Thus

1. $S \wedge T = T \wedge S$ if either s or T has even degree;
2. $S \wedge T = -T \wedge S$ if both S and T have odd degree;
3. $S \wedge S = 0$ if S has odd degree, since $S \wedge S = -S \wedge S$;
4. $T_1 \wedge T_2 \wedge \cdots \wedge S \wedge \cdots \wedge S \wedge \cdots \wedge T_k = 0$ if S has odd degree;
5. If $x_1, \ldots, x_r \in M$ and $i_1, \ldots, i_r$ are selected from $\{1, 2, \ldots, r\}$ then

$$x_{i_1} \wedge \cdots \wedge x_{i_r} = \epsilon_{i_1 \ldots i_r} x_1 \wedge x_2 \wedge \cdots \wedge x_r,$$

where

$$\epsilon_{i_1 \ldots i_r} = \left\{ \begin{array}{ll} 1 & \text{if } i_1, \ldots, i_r \text{ is an even permutation of } 1, \ldots, r; \\ -1 & \text{if } i_1, \ldots, i_r \text{ is an odd permutation of } 1, \ldots, r; \\ 0 & \text{otherwise} \end{array} \right\} = \epsilon^{i_1 \ldots i_r}$$

is called a *permutation symbol*;

6. If $x_j = \alpha^i_j y_i,\ (\alpha^i_j) = A \in K^{r \times r}$ then

$$\begin{aligned} x_1 \wedge \cdots \wedge x_r &= (\alpha^{i_1}_1 y_{i_1}) \wedge \cdots \wedge (\alpha^{i_r}_r y_{i_r}) \\ &= \alpha^{i_1}_1 \ldots \alpha^{i_r}_r\, y_{i_1} \wedge \cdots \wedge y_{i_r} \\ &= \epsilon_{i_1 \ldots i_r} \alpha^{i_1}_1 \ldots \alpha^{i_r}_r\, y_1 \wedge \cdots \wedge y_r \\ &= (\det A) y_1 \wedge \cdots \wedge y_r. \end{aligned}$$

Theorem 9.5. *Let M be n-dimensional, with basis $u_1, \ldots, u_n$. Then*

(i) if $r > n$ then $M^{(r)} = \{0\}$,

(ii) if $1 \le r \le n$ then

$$\dim M^{(r)} = \frac{n!}{r!(n-r)!},$$

and $\{u^{i_1} \wedge \cdots \wedge u^{i_r}\}_{i_1 < \cdots < i_r}$ is a basis for $M^{(r)}$.

Proof ▶

(i) If $r > n$ and $T \in M^{(r)}$ has components $\alpha_{i_1 \dots i_r}$ then the indices $i_1, \dots, i_r$ cannot be distinct. Therefore $T = 0$.

(ii) We have

$$\langle u^i, u_j \rangle = \left\{ \begin{array}{ll} 1 & i = j; \\ 0 & i \neq j \end{array} \right\} = \delta^i_j.$$

More generally:

$$\begin{aligned}
&u^{i_1} \wedge \dots \wedge u^{i_r}[u_{j_1}, \dots, u_{j_r}] \\
&= \sum_{\sigma \in \mathcal{S}_r} \epsilon^\sigma u^{i_1} \otimes \dots \otimes u^{i_r}[u_{j_{\sigma(1)}}, \dots, u_{j_{\sigma(r)}}] \\
&= \sum_{\sigma \in \mathcal{S}_r} \epsilon^\sigma \delta^{i_1}_{j_{\sigma(1)}} \dots \delta^{i_r}_{j_{\sigma(r)}} \\
&= \begin{cases} 1 & \text{if } i_1, \dots, i_r \text{ are distinct and an even permutation of } j_1, \dots, j_r; \\ -1 & \text{if } i_1, \dots, i_r \text{ are distinct and an odd permutation of } j_1, \dots, j_r; \\ 0 & \text{otherwise} \end{cases} \\
&= \delta^{i_1 \dots i_r}{}_{j_1 \dots j_r} \quad (\textit{general Kronecker delta}).
\end{aligned}$$

It follows that if $1 \leq r \leq n$, and if $T \in M^{(r)}$ has components $\alpha_{i_1 \dots i_r}$ then the tensor:

$$(*) \qquad \sum_{i_1 < \dots < i_r} \alpha_{i_1 \dots i_r} u^{i_1} \wedge \dots \wedge u^{i_r}$$

has components

$$\begin{aligned}
\sum_{i_1 < \dots < i_r} \alpha_{i_1 \dots i_r} u^{i_1} \wedge \dots \wedge u^{i_r}[u_{j_1}, \dots, u_{j_r}] &= \sum_{i_1 < \dots < i_r} \alpha_{i_1 \dots i_r} \delta^{i_1 \dots i_r}{}_{j_1 \dots j_r} \\
&= \alpha_{j_1 \dots j_r},
\end{aligned}$$

provided $j_1 < \dots < j_r$. Therefore $(*)$ has the same components as T. Therefore

$$(**) \quad \{u^{i_1} \wedge \dots \wedge u^{i_r}\}_{i_1 < \dots < i_r}$$

generate $M^{(r)}$. Also

$$(*) = 0 \Rightarrow \alpha_{j_1 \dots j_r} = 0.$$

Therefore $(**)$ are linearly independent. Therefore $(**)$ form a basis for $M^{(r)}$. ◀

Chapter 10

Classification of Linear Operators

10.1 Hamilton-Cayley and Primary Decomposition

Let $M \xrightarrow{T} M$ be a linear operator on a vector space M over a field K. Let $K[X]$ be the ring of polynomials in X with coefficients in K. If $p = \alpha_0 + \alpha_1 X + \alpha_2 X^2 + \cdots + \alpha_r X^r$, write

$$p(T) = \alpha_0 1 + \alpha_1 T + \alpha_2 T^2 + \cdots + \alpha_r T^r \in \mathcal{L}(M).$$

Theorem 10.1 (Hamilton-Cayley). *Let $T \in \mathcal{L}(M)$ have characteristic polynomial p, and let $\dim M < \infty$. Then $p(T) = 0$.*

Proof ▶ Let T have matrix α^i_j wrt basis u_i. Put $P = (p^i_j)$, where $p^i_j = \alpha^i_j - X\delta^i_j$. Then P is an $n \times n$ matrix of polynomials, and $\det P = p$ is the characteristic polynomial.

Let $q^i_j = (-1)^{i+j}$ times the determinant of the matrix got from P by removing the i^{th} column and j^{th} row. Then $Q = (q^i_j)$ is also an $n \times n$ matrix of polynomials, and

$$PQ = (\det P)I,$$

i.e.

$$p^i_k q^k_j = p\delta^i_j$$

Therefore

$$p(T)u_j = p(T)\delta^i_j u_i = p^i_k(T)q^k_j(T)u_i$$
$$= q^k_j(T)[\alpha^i_k - T\delta^i_k]u_i = q^k_j(T)[\alpha^i_k u_i - Tu_k] = 0$$

Therefore $p(T) = 0$, as required. ◄

Example: $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$.

$$p = \begin{vmatrix} \alpha - X & \beta \\ \gamma & \delta - X \end{vmatrix} = X^2 - (\alpha + \delta)X + \alpha\delta - \beta\gamma.$$

Therefore

$$\begin{aligned} p\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^2 - (\alpha + \delta)\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} + (\alpha\delta - \beta\gamma)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Theorem 10.2 (Primary Decomposition Theorem). *Let $T \in \mathcal{L}(M)$, and let*

$$(T - \lambda_1 1)^{r_1} \ldots (T - \lambda_k 1)^{r_k} = 0,$$

where $\lambda_1, \ldots, \lambda_k$ are distinct scalars, and $r_1, \ldots, r_k$ are positive integers. Then

$$M = M_1 \oplus \cdots \oplus M_k,$$

where $M_i = \ker(T - \lambda_i 1)^{r_i}$ for $i = 1, \ldots, k$.

Proof ► Let

$$\begin{aligned} f &= (X - \lambda_1)^{r_1} \ldots (X - \lambda_k)^{r_k}, \\ g_i &= (X - \lambda_i)^{r_i}, \\ f &= g_i h_i \end{aligned}$$

(say), so $f(T) = 0$ and $M_i = \ker g_i(T)$.

Now $h_1, \ldots, h_k$ have hcf 1. Therefore there exist

$$\Theta_1, \ldots, \Theta_k \in K[X]$$

such that

$$\Theta_1 h_1 + \cdots + \Theta_k h_k = 1.$$

Put $P_i = \Theta_i(T) h_i(T)$. Then

(i) $P_1 + \cdots + P_k = 1$. Also:

(ii) for each $x \in M$,

$$g_i(T)P_i x = g_i(T)\Theta_i(T)h_i(T)x = f(T)\Theta_i(T)x = 0.$$

Therefore $P_i x \in M_i$,

(iii) if $x_i \in M_i$ and $j \neq i$ then

$$P_j x_i = \Theta_j(T)h_j(T)x_i = 0,$$

since g_i is a factor of h_j, and $g_i(T)x_i = 0$.

Thus

1. for each $x \in M$ we have

$$x = 1x = P_1 x + \cdots + P_k x, \quad P_i x \in M_i;$$

2. if $x = x_1 + \cdots + x_k$, with $x_i \in M_i$, then (for example, see Figure 10.1)

$$x_i = (P_1 + \cdots + P_k)x_i = P_i x_i = P_i(x_1 + \cdots + x_k) = P_i x.$$

Therefore x_i is uniquely determined by x. Therefore

$$M = M_1 \oplus \cdots \oplus M_k,$$

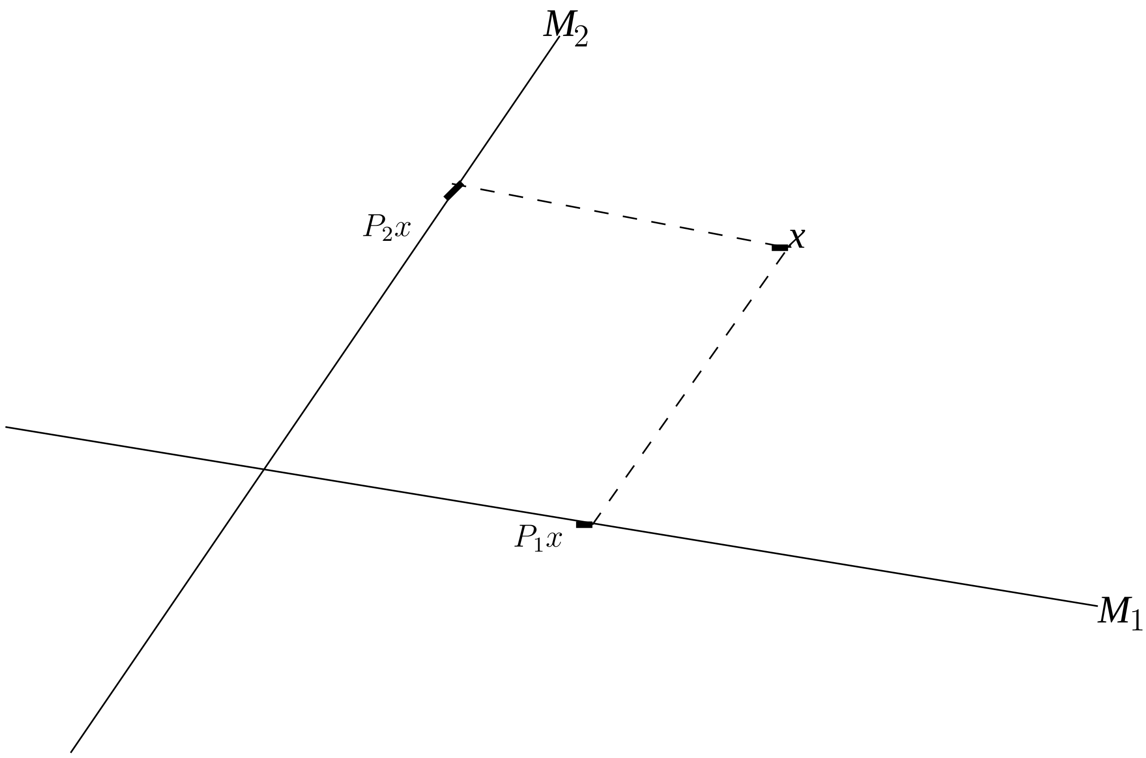

Figure 10.1

as required. ◀

Note. Each subspace M_i is invariant under T, since

$$\begin{aligned} x \in M_1 &\Rightarrow g_i(T)x = 0 \\ &\Rightarrow T g_i(T)x = 0 \\ &\Rightarrow g_i(T)Tx = 0 \\ &\Rightarrow Tx \in M_i. \end{aligned}$$

Therefore, if we take bases for $M_1, \ldots, M_k$, and put them together to get a basis for M, then wrt this basis T has matrix

$$\begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_k \end{pmatrix},$$

where A_i is the matrix of T_{M_i}, the restriction of T to M_i.

Note also that

$$(T_{M_i} - \lambda_i 1)^{r_i} = 0.$$

Example: Let $M \xrightarrow{T} M$ and $T^2 = T$ (T a *projection operator*). Then

$$T(T-1) = 0.$$

Therefore, by primary decomposition,

$$\begin{aligned} M &= \ker(T-1) \oplus \ker T \\ &= \text{1-eigenspace} \oplus \text{0-eigenspace}. \end{aligned}$$

If T has rank r, and $u_1, \dots, u_r$ is basis of 1-eigenspace; $u_{r+1}, \dots, u_n$ is basis of 0-eigenspace then, wrt $u_1, \dots, u_n$ T has matrix

$$\left(\begin{array}{cc} \boxed{\begin{matrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{matrix}} & 0 \\ 0 & \boxed{\begin{matrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{matrix}} \end{array} \right) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

10.2 Diagonalisable Operators

Let $M \xrightarrow{T} M$; $\dim M < \infty$. Then:

$(T - \lambda_1 1) \dots (T - \lambda_k 1) = 0 \quad (\lambda_1, \dots, \lambda_k \text{ distinct})$
$\Rightarrow M = \ker(T - \lambda_1 1) \oplus \dots \oplus (T - \lambda_k 1)$ by Primary Decomposition
$\Rightarrow M = (\lambda_1\text{-eigenspace}) \oplus \dots \oplus (\lambda_k\text{-eigenspace})$
$\Rightarrow T$ has a diagonal matrix wrt some basis of M
$\Rightarrow M$ has basis consisting of eigenvectors of T; and $(T - \lambda_1 1) \dots (T - \lambda_k 1)u = 0$ for each eigenvector u, where $\lambda_1, \dots, \lambda_k$ are the distinct eigenvalues of T
$\Rightarrow (T - \lambda_1 1) \dots (T - \lambda_k 1) = 0 \quad (\lambda_1, \dots, \lambda_k \text{ distinct})$.

Definition. A linear operator T with any one (and hence all) of the above properties is called *diagonalisable.*

Example: The operator

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$$

is not diagonalisable.

Proof of This ▷ The characteristic polynomial is

$$\begin{vmatrix} 1-X & 0 \\ 1 & 1-X \end{vmatrix} = (1-X)^2.$$

Therefore 1 is the only eigenvalue.

Also

$$\begin{aligned}(\alpha,\beta) \in \text{1-eigenspace} &\Leftrightarrow \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &\Leftrightarrow \begin{pmatrix} \alpha \\ \alpha+\beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &\Leftrightarrow \alpha = 0.\end{aligned}$$

Therefore 1-eigenspace $= \{\beta(0,1) : \beta \in \mathbb{R}\}$ is 1-dimensional. Therefore $\mathbb{R}^2$ does not have a basis of eigenvectors of $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. ◁

Theorem 10.3. *Let $S, T \in \mathcal{L}(M)$ be diagonalisable* ($\dim M < \infty$)*. Then there exists a basis wrt which both S and T have diagonal matrices (S, T* simultaneously diagonalisable*) iff $ST = TS$ (S, T commute).*

Proof ▶

(i) Let M have a basis wrt which S has diagonal matrix

$$A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

and T has diagonal matrix

$$B = \begin{pmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_n \end{pmatrix}$$

Then $AB = BA$. Therefore $ST = TS$.

(ii) Let $ST = TS$. Since S is diagonalisable we have:

$$M = M_1 \oplus \cdots \oplus M_i \oplus \cdots \oplus M_k,$$

distinct sum of eigenspaces of S. Since S and T commute, T leaves each M_i invariant:

$$T_{M_i} : M_i \to M_i.$$

Since T is diagonalisable we have:

$$(T - \mu_1 1) \dots (T - \mu_l 1) = 0,$$

distinct $\mu_1, \dots, \mu_l$. Therefore

$$(T_{M_i} - \mu_1 1_{M_i}) \dots (T_{M_i} - \mu_l 1_{M_i}) = 0.$$

Therefore T_{M_i} is diagonalisable. Therefore M_i has a basis of eigenvectors of T. Therefore M has a basis of eigenvectors of S, and of T. ◄

10.3 Conjugacy Classes

Problem: Given two linear operators

$$S, T : M \to M,$$

to determine whether they are equivalent up to an isomorphism of M, i.e. is there a linear isomorphism $M \xrightarrow{R} M$ so that the diagram

$$\begin{array}{ccc} M & \xrightarrow{S} & M \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle R} \\ M & \xrightarrow[T]{} & M \end{array}$$

commutes, i.e.

$$RS = TR,$$

i.e.

$$RSR^{-1} = T?$$

Definition. S is *conjugate to* T if there exists a linear isomorphism R such that

$$RSR^{-1} = T.$$

Conjugacy is an equivalence relation on $\mathcal{L}(M)$; the equivalence classes are called *conjugacy classes*.

If $T \in \mathcal{L}(M)$ has matrix $A \in K^{n\times n}$ wrt some basis of M then the set of all matrices which can represent T is:

$$\{PAP^{-1} : P \in K^{n\times n} \text{ is invertible}\},$$

which is a conjugacy class in $K^{n\times n}$.

Conversely, the set of all linear operators on M which can be represented by A is:

$$\{RTR^{-1} : R \text{ is a linear isomorphism of } M\},$$

which is a conjugacy class in $\mathcal{L}(M)$.

Hence we have a bijective map from the set of conjugacy classes in $\mathcal{L}(M)$ to the set of conjugacy classes in $K^{n\times n}$ (see Figure 10.2).

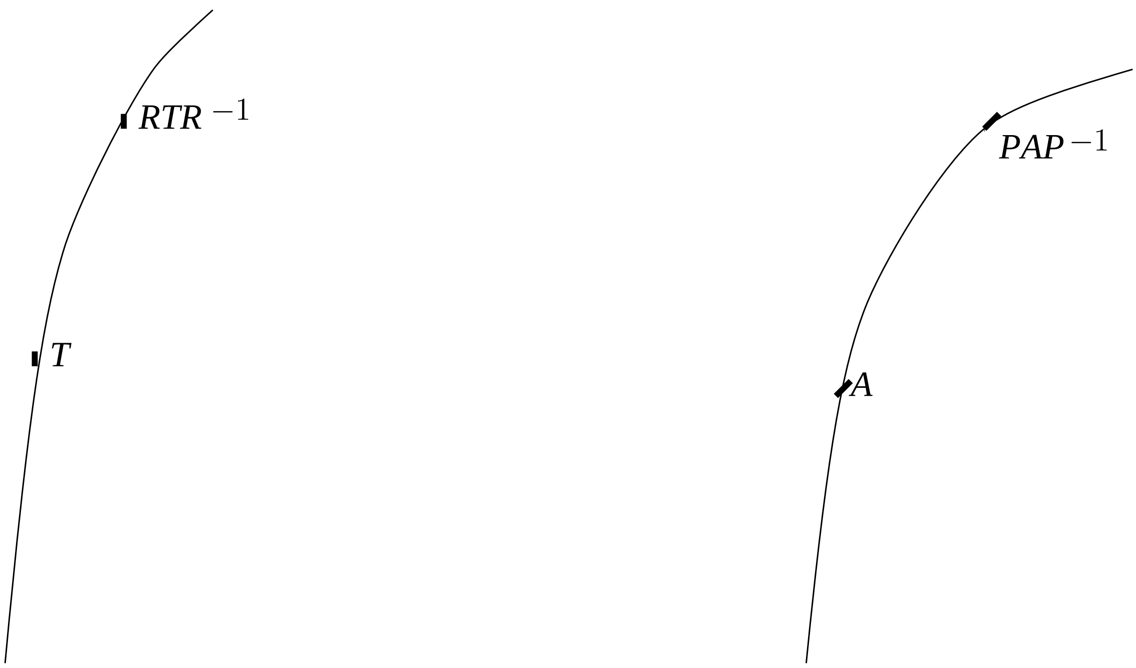

Figure 10.2

The problem of determining which conjugacy class T belongs to is thus equivalent to determining which conjugacy class A belongs to.

A simple way of distinguishing conjugacy classes is to use properties such as: rank, trace, determinant, eigenvalues, characteristic polynomial, which are the same for all elements of a conjugacy class.

Examples:

1. Let

$$J = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 1 & \lambda & 0 & 0 \\ 0 & 1 & \lambda & 0 \\ 0 & 0 & 1 & \lambda \end{pmatrix}, \quad \text{a } \textit{Jordan } \lambda\textit{-block of size } 4.$$

(4×4, λ on diagonal, 1 just below diagonal, zero elsewhere).

$$J - \lambda I = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Therefore

$$\begin{aligned} (J - \lambda I)e_1 &= e_2, \\ (J - \lambda I)e_2 &= e_3, \\ (J - \lambda I)e_3 &= e_4, \\ (J - \lambda I)e_4 &= 0. \end{aligned}$$

Thus

$$\begin{array}{ccccccc} K^4 & & e_1 & e_2 & e_3 & e_4 \\ \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow \\ \mathrm{im}(J - \lambda I) & & e_2 & e_3 & e_4 & 0 \\ \downarrow & & \downarrow & \downarrow & \downarrow & \\ \mathrm{im}(J_\lambda I)^2 & & e_3 & e_4 & 0 & \\ \downarrow & & \downarrow & \downarrow & & \\ \mathrm{im}(j - \lambda I)^3 & & e_4 & 0 & & \\ \downarrow & & \downarrow & & & \\ \{0\} & & 0 & & & \end{array}$$

Thus

$$\begin{aligned} &\mathrm{im}(J - \lambda I) \text{ has basis } e_2, e_3, e_4, \ \mathrm{rank}(J - \lambda I) = 3, \\ &\mathrm{im}(J - \lambda I)^2 \text{ has basis } e_3, e_4, \ \mathrm{rank}(J - \lambda I)^2 = 2, \\ &\mathrm{im}(J - \lambda I)^3 \text{ has basis } e_4, \ \mathrm{rank}(J - \lambda I)^3 = 1, \\ &(J - \lambda I)^4 = 0. \end{aligned}$$

$$\mathrm{char}\, J = \begin{vmatrix} \lambda - X & 0 & 0 & 0 \\ 1 & \lambda - X & 0 & 0 \\ 0 & 1 & \lambda - X & 0 \\ 0 & 0 & 1 & \lambda - X \end{vmatrix} = (\lambda - X)^4.$$

Therefore λ is the only eigenvalue of J, and the λ-eigenspace $= \ker(J - \lambda I)$ has basis e_4.

2. Let

$$J = \begin{pmatrix} \boxed{\begin{matrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{matrix}} & 0 & 0 \\ 0 & \boxed{\begin{matrix} \lambda & 0 \\ 1 & \lambda \end{matrix}} & 0 \\ 0 & 0 & \boxed{\begin{matrix} \lambda & 0 \\ 1 & \lambda \end{matrix}} \end{pmatrix}$$

(Jordan λ-blocks on diagonal: $3 \times 3, 2 \times 2, 2 \times 2$).

$$\begin{array}{cllll} K^7 & & \underbrace{e_1}_{s_1} & \underbrace{e_2 e_4 e_6}_{s_2} & \underbrace{e_3 e_5 e_7}_{s_3} \\ \downarrow & & & & \\ \mathrm{im}(J - \lambda I) & & \underbrace{e_2}_{s_1} & \underbrace{e_3 e_5 e_7}_{s_2} & \\ \downarrow & & & & \\ \mathrm{im}(J - \lambda I)^2 & & \underbrace{e_3}_{s_1} & & \\ \downarrow & & & & \\ \{0\} & & & & \end{array}$$

where s_1, s_2 and s_3 are the dimensions of the kernel of $(J - \lambda I)$ restricted to $\mathrm{im}(J - \lambda I)^2, \mathrm{im}(J - \lambda I)$ and K^7 respectively.

char $J = (\lambda - X)^7$. λ is the only eigenvector; dim λ-eigenspace $= 3 =$ number of Jordan blocks.

$$\begin{aligned} (J - \lambda)e_1 &= e_2 \\ (J - \lambda)e_2 &= e_3 \\ (J - \lambda)e_3 &= 0 \quad \text{eigenvector} \\ \\ (J - \lambda)e_4 &= e_5 \\ (J - \lambda)e_5 &= 0 \quad \text{eigenvector} \\ \\ (J - \lambda)e_6 &= e_7 \\ (J - \lambda)e_7 &= 0 \quad \text{eigenvector.} \end{aligned}$$

10.4 Jordan Forms

Definition. A square matrix $J \in K^{n \times n}$ is called a *Jordan matrix* if it is of the form

$$J = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_l \end{pmatrix},$$

where each J_i is a Jordan block.

Example:

$$J = \begin{pmatrix} \boxed{\begin{matrix} \lambda & 0 & 0 \\ 1 & \lambda & 0 \\ 0 & 1 & \lambda \end{matrix}} & & & \\ & \boxed{\begin{matrix} \lambda & 0 \\ 1 & \lambda \end{matrix}} & & \\ & & \boxed{\begin{matrix} \mu & 0 \\ 1 & \mu \end{matrix}} & \\ & & & \boxed{\begin{matrix} \mu & 0 \\ 1 & \mu \end{matrix}} \end{pmatrix},$$

(where $\lambda \neq \mu$ (say)) is a 9×9 Jordan matrix.

Note that

(i) $\operatorname{char} J = (\lambda - X)^5(\mu - X)^4$; eigenvalue λ, with *algebraic multiplicity* 5; eigenvalue μ, with algebraic multiplicity 4.

(ii)

$$\begin{aligned} \text{dimension of } \lambda\text{-eigenspace} &= \text{number of } \lambda\text{-blocks} \\ &= \textit{geometric multiplicity} \text{ of eigenvalue } \lambda = 2; \end{aligned}$$

geometric multiplicity of eigenvalue $\mu = 2$.

(iii)

$$J - \lambda I = \begin{pmatrix} \boxed{\begin{matrix}0&0&0\\1&0&0\\0&1&0\end{matrix}} & & & \\ & \boxed{\begin{matrix}0&0\\1&0\end{matrix}} & & \\ & & \boxed{\begin{matrix}\mu-\lambda & 0\\ 1 & \mu-\lambda\end{matrix}} & \leftarrow \text{non-sing.} \\ & & \text{non-sing.} \rightarrow & \boxed{\begin{matrix}\mu-\lambda & 0\\ 1 & \mu-\lambda\end{matrix}} \end{pmatrix},$$

$$(J - \lambda I)^2 = \begin{pmatrix} \boxed{\begin{matrix}0&0&0\\0&0&0\\1&0&0\end{matrix}} & & & \\ & \boxed{\begin{matrix}0&0\\0&0\end{matrix}} & & \\ & & \boxed{\begin{matrix}(\mu-\lambda)^2 & 0\\ 1 & (\mu-\lambda)^2\end{matrix}} & \leftarrow \text{non-sing.} \\ & & \text{non-sing.} \rightarrow & \boxed{\begin{matrix}(\mu-\lambda)^2 & 0\\ 1 & (\mu-\lambda)^2\end{matrix}} \end{pmatrix},$$

$$(J - \lambda)^3 = \begin{pmatrix} \boxed{\begin{matrix}0&0&0\\0&0&0\\0&0&0\end{matrix}} & & & \\ & \boxed{\begin{matrix}0&0\\0&0\end{matrix}} & & \\ & & \boxed{\text{N.S.}} & \\ & & & \boxed{\text{N.S.}} \end{pmatrix}.$$

Therefore

$$\begin{aligned} \operatorname{rank}(J - \lambda I) &= 2 + 1 + 4, \\ \operatorname{rank}(J - \lambda I)^2 &= 1 + 0 + 4, \\ \operatorname{rank}(J - \lambda I)^3 &= 0 + 0 + 4. \end{aligned}$$

More generally, if J is a Jordan $n \times n$ matrix, with

$$\begin{aligned} &b_1 \ \lambda\text{-blocks of size } 1,\\ &b_2 \ \lambda\text{-blocks of size } 2,\\ &\quad\vdots\\ &b_k \ \lambda\text{-blocks of size } k, \end{aligned}$$

and if λ has algebraic multiplicity m then

$$b_1 + 2b_2 + 3b_3 + \cdots = m,$$

$\operatorname{rank}(J - \lambda I)$ has rank:

$$0b_1 + 1b_2 + 2b_3 + 3b_4 + \cdots + (n - m),$$

$\operatorname{rank}(J - \lambda I)^2$ has rank:

$$0b_1 + 0b_2 + 1b_3 + 2b_4 + \cdots + (n - m),$$

$\operatorname{rank}(J - \lambda I)^3$ has rank:

$$0b_1 + 0b_2 + 0b_3 + 1b_4 + 2b_5 + \cdots + (n - m),$$

and so on. Hence the number b_k of λ-blocks of size k in J is uniquely determined by the conjugacy class of J.

Theorem 10.4. *Let $T \in \mathcal{L}(M)$ be a linear operator on a finite dimensional vector space over a field K which is algebraically closed. Then T can be represented by a Jordan matrix J. The matrix J, which by the preceding is uniquely determined, apart from the arrangement of the blocks on the diagonal, is called the* Jordan form *of T.*

Proof ▶ Since K is algebraically closed, the characteristic polynomial is a product of linear factors; so, by Hamilton-Cayley we have

$$(T - \lambda_1 1)^{r_1} \ldots (T - \lambda_k 1)^{r_k} = 0$$

(say), with $\lambda_1, \ldots, \lambda_k$ distinct factors.

By primary decomposition:

$$M = M_1 \oplus \cdots \oplus M_k,$$

where $M_i = \ker(T - \lambda_i 1)^{r_i}$. We will show that M_i has a basis wrt which T_{M_i} has a Jordan matrix with λ_i on the diagonal.

Put $S = T_{M_i} - \lambda_i 1_{M_i}$. Then

$$M_i \xrightarrow{S} M_i$$

and $S^{r_i} = 0$, i.e. S is a *nilpotent operator*. Suppose $S^r = 0$ but $S^{r-1} \neq 0$, and consider:

$$\begin{array}{cl}
M_i & a_1 \dots a_{s_1} b_1 \dots b_{s_2} \dots y_1 \dots y_{s_{r+1}} z_1 \dots z_{s_r} \\
\downarrow & \\
\operatorname{im} S & \\
\downarrow & \\
\operatorname{im} S^2 & \\
\downarrow & \\
\vdots & \\
\downarrow & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (*) \\
\operatorname{im} S^{r-3} & x_1 \dots x_{s_1} y_1 \dots y_{s_2} z_1 \dots z_{s_1} \dots z_{s_2} \dots z_{s_3} \\
\downarrow & \\
\operatorname{im} S^{r-2} & y_1 \dots y_{s_1} z_1 \dots z_{s_1} \dots z_{s_2} \\
\downarrow & \\
\operatorname{im} S^{r-1} & z_1 \dots z_{s_1} \\
\downarrow & \\
\{0\} &
\end{array}$$

Choose a basis $z_1, \dots, z_{s_1}$ for $\operatorname{im} S^{r-1}$. Choose $y_1, \dots, y_{s_1} \in \operatorname{im} S^{r-2}$ such that $Sy_j = z_j$. Extend to a basis $z_1, \dots, z_{s_1}, \dots, z_{s_2}$ for the kernel of $\operatorname{im} S^{r-2} \to \operatorname{im} S^{r-1}$. Thus $y_1, \dots, y_{s_1}, z_1, \dots, z_{s_2}$ is a basis for $\operatorname{im} S^{r-2}$.

Now repeat the construction: choose $x_1, \dots, x_{s_1}, y_1, \dots, y_{s_2} \in \operatorname{im} S^{r-3}$ such that $Sx_j = y_j$, $Sy_i = z_i$. Extend to a basis $z_1, \dots, z_{s_2}, \dots, z_{s_3}$ for the kernel of $\operatorname{im} S^{r-3} \to \operatorname{im} S^{r-2}$. Thus $x_1, \dots, x_{s_1}, y_1, \dots, y_{s_2}, z_1, \dots, z_{s_3}$ is a basis for $\operatorname{im} S^{r-3}$.

Continue in this way until we get a basis

$$a_1, \dots, a_{s_1}, b_1, \dots, b_{s_2}, \dots, y_1, \dots, y_{s_{r-1}}, z_1, \dots, z_{s_r}$$

(say), for M_i, with

$$Sa_j = b_j, Sb_j = c_j, \dots, Sy_j = z_j, Sz_j = 0.$$

Now write the basis elements in order, by going down each column of $(*)$ in turn, starting at the left most column (and leaving out any column whose elements have already been written down).

Relative to this basis for M_i the matrix of S is a Jordan matrix with zeros on the diagonal. Therefore the matrix of $T_{M_i} = S + \lambda_i 1_{M_i}$ is a Jordan matrix with λ_i on the diagonal.

Putting together these bases for $M_1, \ldots, M_k$ we get a basis for M wrt which T has a Jordan matrix, as required. ◀

Example: To find a Jordan matrix J conjugate to the matrix

$$A = \begin{pmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{pmatrix}.$$

The characteristic polynomial is

$$\begin{aligned} p &= \begin{vmatrix} 5-X & 4 & 3 \\ -1 & -X & -3 \\ 1 & -2 & 1-X \end{vmatrix} \\ &= (5-X)[-X(1-X)-6] - 4[-(1-X)+3] + 3[2+X] \\ &= (5-X)[X^2-X-6] - 4[X+2] + 3[2+X] \\ &= 5X^2 - 5X - 30 - X^3 + X^2 + 6X - 4X - 8 + 3X + 6 \\ &= -X^3 + 6X^2 - 32 \\ &= (X+2)(-X^2+8X-16) \\ &= -(X+2)(X-4)^2. \end{aligned}$$

Therefore, by Hamilton-Cayley the operator $\mathbb{R}^3 \xrightarrow{A} \mathbb{R}^3$ satisfies:

$$(A+2I)(A-4I)^2 = 0.$$

Therefore, by primary decomposition:

$$\mathbb{R}^3 = \ker(A+2I) \oplus \ker(A-4I)^2.$$

Now

$$\begin{aligned} \ker(A+2I) &= \ker \begin{pmatrix} 7 & 4 & 3 \\ -1 & 2 & -3 \\ 1 & -2 & 3 \end{pmatrix} \\ &= \ker \begin{pmatrix} 7 & 4 & 3 \\ 0 & 18 & -18 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{matrix} \text{row 3} + \text{row 2} \\ 7\,\text{row 2} + \text{row 1} \end{matrix} \end{aligned}$$

Therefore

$$\begin{aligned}(\alpha,\beta,\gamma)\in\ker(A+2I) &\Leftrightarrow 7\alpha+4\beta+3\gamma=0\\ &\qquad \beta-\gamma=0\\ &\Leftrightarrow 7\alpha+7\gamma=0\\ &\qquad \beta-\gamma=0\\ &\Leftrightarrow (\alpha,\beta,\gamma)=(\alpha,-\alpha,-\alpha)=\alpha(1,-1,-1).\end{aligned}$$

Therefore $\ker(A+2I)$ has basis $u_1=(1,-1,-1)$.

$$\begin{aligned}\ker(A-4I)^2 &= \ker\begin{pmatrix}1 & 4 & 3\\ -1 & -4 & -3\\ 1 & -2 & -3\end{pmatrix}^2\\ &= \ker\begin{pmatrix}0 & -18 & -18\\ 0 & 18 & 18\\ 0 & 18 & 18\end{pmatrix}\\ &= \ker\begin{pmatrix}0 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\end{aligned}$$

Therefore

$$\begin{aligned}(\alpha,\beta,\gamma)\in\ker(A-4I)^2 &\Leftrightarrow \beta+\gamma=0\\ &\Leftrightarrow (\alpha,\beta,\gamma)=(\alpha,\beta,-\beta)=\alpha(1,0,0)+\beta(0,1,-1).\end{aligned}$$

Therefore $(1,0,0),(0,1,-1)$ is a basis for $\ker(A-4I)^2$.

Put

$$u_2=(1,0,0),\quad u_3=(A-4I)u_2=(1,-1,1).$$

So u_1,u_2,u_3 is a basis for $\mathbb{R}^3$ such that

$$\begin{aligned}(A+2I)u_1&=0,\\ (A-4I)u_2&=u_3,\\ (A-4I)u_3&=0,\end{aligned}$$

i.e.

$$\begin{aligned}Au_1&=-2u_1,\\ Au_2&=4u_2+u_3,\\ Au_3&=4u_3.\end{aligned}$$

Therefore wrt basis

$$\begin{aligned}
u_1 &= (1,-1,-1) = e_1 - e_2 - e_3, \\
u_2 &= (1,0,0) = e_1, \\
u_3 &= (1,-1,1) = e_1 - e_2 + e_3
\end{aligned}$$

the operator A has matrix

$$J = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 1 & 4 \end{pmatrix}$$

Let P be the transition matrix from (e_1, e_2, e_3) to (u_1, u_2, u_3).

$$P = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 0 & -1 & -1 \\ 2 & 2 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & 1 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$

Therefore

$$\begin{aligned}
PAP^{-1} &= \begin{pmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & 1 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 1 \\ 4 & 4 & 0 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 1 & 4 \end{pmatrix} = J,
\end{aligned}$$

as required.

Note. (i)

$$\left.\begin{aligned} u_1 &= e_1 - e_2 - e_3 \\ u_2 &= e_1 \\ u_3 &= e_1 - e_2 + e_3 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} e_1 &= u_2 \\ u_1 + u_3 &= 2e_1 - 2e_2 \\ u_1 - u_3 &= -2e_3 \end{aligned}\right\}$$

$$\Rightarrow \left.\begin{aligned} e_1 &= u_2 \\ e_2 &= -\tfrac{1}{2}u_1 + u_2 - \tfrac{1}{2}u_3 \\ e_3 &= -\tfrac{1}{2}u_1 + \tfrac{1}{2}u_3 \end{aligned}\right\}$$

$$\Rightarrow P = \begin{pmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & 1 & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

(ii)

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & -1 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 2 & 2 & 0 & 2 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 & -1 & 1 \end{pmatrix}$$

Example: To find the Jordan form of

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix}.$$

$$\begin{aligned}
\operatorname{char} A &= \begin{vmatrix} 1-X & 1 & 3 \\ 5 & 2-X & 6 \\ -2 & -1 & -3-X \end{vmatrix} \\
&= (1-X)[(2-X)(-3-X)+6] - [5(-3-X)+12] + 3[-5+2(2-X)] \\
&= (1-X)[X^2+X] - [-5X-3] + 3[-1-2X] \\
&= X^2 + X - X^3 - X^2 + 5X + 3 - 6X - 3 \\
&= -X^3.
\end{aligned}$$

Therefore operator $\mathbb{R}^3 \xrightarrow{A} \mathbb{R}^3$ satisfies

$$A^3 = 0.$$

Now

$$A^2 = \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{pmatrix}.$$

So put

$$\begin{aligned}
u_1 &= e_1 = (1,0,0) = e_1, \\
u_2 &= Ae_1 = (1,5,-2) = e_1 + 5e_2 - 2e_3, \\
u_3 &= A^2 e_1 = (0,3,-1) = 3e_2 - e_3.
\end{aligned}$$

So wrt new basis u_1, u_2, u_3 operator A has matrix

$$J = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Also

$$u_2 - 2u_3 = e_1 - e_2.$$

So

$$\begin{aligned} e_2 &= u_1 - u_2 + 2u_3, \\ e_3 &= 3e_2 - u_3 = 3u_1 - 3u_2 + 5u_3. \end{aligned}$$

Thus

$$\begin{aligned} e_1 &= u_1, \\ e_2 &= u_1 - u_2 + 2u_3, \\ e_3 &= 3u_1 - 3u_2 + 5u_3. \end{aligned}$$

Therefore $PAP^{-1} = J$, where

$$P = \begin{pmatrix} 1 & 1 & 3 \\ 0 & -1 & -3 \\ 0 & 2 & 5 \end{pmatrix}.$$

Check:

$$\begin{aligned} PA = JP \Leftrightarrow \begin{pmatrix} 1 & 1 & 3 \\ 0 & -1 & -3 \\ 0 & 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 3 \\ 0 & -1 & -3 \\ 0 & 2 & 5 \end{pmatrix} \\ \Leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 3 \\ 0 & -1 & -3 \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 3 \\ 0 & -1 & -3 \end{pmatrix}. \end{aligned}$$

10.5 Determinants

Let M be a vector space of finite dimension n, and $M \xrightarrow{T} N$ be a linear operator. The *pull-back* (or *transpose*) of T is the operator

$$M^* \xleftarrow{T^*} N^*$$

defined by

$$\langle T^* f, x \rangle = \langle f, Tx \rangle.$$

The transpose of T^* is T itself (by duality) written T_*. So:

$$\begin{aligned} M &\xrightarrow{T_*} N \quad (T_* = T), \\ M^* &\xleftarrow{T^*} N^*, \\ \langle T^* f, x \rangle &= \langle f, T_* x \rangle. \end{aligned}$$

If $M = N$ and T has matrix $A = (\alpha^i_j)$ wrt basis u_i then

$$\langle T^* u^j, u_i \rangle = \langle u^j, T_* u_i \rangle = \langle u^j, \alpha^k_i u_k \rangle = \alpha^j_i.$$

Therefore T^* has matrix A^t wrt basis u^i.

More generally we have the pull-back

$$M^{(r)} \overset{T^*}{\longleftarrow} N^{(r)},$$

and push-forward

$$M_{(r)} \overset{T_*}{\longrightarrow} N_{(r)}$$

defined by

$$\begin{aligned}(T^* S)(x_1, \ldots, x_r) &= S(T_* x_1, \ldots, T_* x_r),\\ (T_* S)(f^1, \ldots, f^r) &= S(T^* f^1, \ldots, T^* f^r).\end{aligned}$$

These maps T^*, T_* are linear, and preserve the wedge-product. In particular the spaces $M^{(n)}$ and $M_{(n)}$ are 1-dimensional. Therefore for $M \overset{T}{\to} M$ the push-forward:

$$M_{(n)} \overset{T_*}{\to} M_{(n)},$$

and the pull-back

$$M^{(n)} \overset{T^*}{\leftarrow} M^{(n)}$$

must each be multiplication by a scalar (called $\det T, \det T^*$ respectively).

To see what these scalars are let T have matrix $A = (\alpha^i_j)$ wrt basis u_i. Then

$$\begin{aligned}T_*(u_1 \wedge \cdots \wedge u_n) &= T u_1 \wedge \cdots \wedge T u_n\\ &= (\alpha^{i_1}_1 u_{i_1}) \wedge \cdots \wedge (\alpha^{i_n}_n u_{i_n})\\ &= \det A \, u_1 \wedge \cdots \wedge u_n.\end{aligned}$$

Therefore $M_{(n)} \overset{T_*}{\to} M_{(n)}$ is multiplication by $\det A$. Similarly $M^{(n)} \overset{T^*}{\leftarrow} M^{(n)}$ is multiplication by $\det A^t = \det A$. Therefore

$$\det T = \det T^* = \det A,$$

independent of choice of basis.

Example: Let $\dim M = 4$, and $M \overset{T}{\to} M$ have matrix $A = (\alpha^i_j)$ wrt basis u_i. Then

$$\dim M_{(2)} = \frac{4!}{2!2!} = 6,$$

and wrt basis $u_1 \wedge u_2, u_1 \wedge u_3, u_1 \wedge u_4, u_2 \wedge u_3, u_2 \wedge u_4, u_3 \wedge u_4$

$$T_* : M_{(2)} \to M_{(2)}$$

satisfies

$$\begin{aligned} T_*(u_1 \wedge u_2) &= Tu_1 \wedge Tu_2 \\ &= (\alpha_1^1 u_1 + \alpha_1^2 u_2 + \alpha_1^3 u_3 + \alpha_1^4 u_4) \wedge (\alpha_2^1 u_1 + \alpha_2^2 u_2 + \alpha_2^3 u_3 + \alpha_2^4 u_4) \\ &= (\alpha_1^1 \alpha_2^2 - \alpha_1^2 \alpha_2^1) u_1 \wedge u_2 + \dots . \end{aligned}$$

Therefore matrix of T_* is a 6×6 matrix whose entries are 2×2 subdeterminants of A.

Theorem 10.5. *If $M \xrightarrow{T} M$ has rank r then*

(i) $M_{(r)} \xrightarrow{T_*} M_{(r)}$ is non-zero,

(ii) $M_{(r+1)} \xrightarrow{T_*} M_{(r+1)}$ is zero.

Proof ▶

(i) Let $y_1, \dots, y_r$ be a basis for $\operatorname{im} T$, and let $y_i = Tx_i$. Then

$$T_* \, x_1 \wedge \dots \wedge x_r = Tx_1 \wedge \dots \wedge Tx_r = y_1 \wedge \dots \wedge y_r \neq 0.$$

(ii) Let u_i be a basis for M. Then

$$T_* \, u_{i_1} \wedge \dots \wedge u_{i_{r+1}} = Tu_{i_1} \wedge \dots \wedge Tu_{i_{r+1}} = 0,$$

since $Tu_{i_1}, \dots, Tu_{i_{r+1}} \in \operatorname{im} T$, which has dimension r. Therefore linearly independent. ◀

Corollary 10.1. *If T has matrix A then* $\operatorname{rank} T = r \Leftrightarrow$ *all $(r+1) \times (r+1)$ subdeterminants are zero, and there exists at least one non-zero $r \times r$ subdeterminant.*

Chapter 11

Orientation

11.1 Orientation of Vector Spaces

Let M be a finite dimensional real vector space. Let $P = (p^i_j)$ be the transition matrix from basis $u_1, \ldots, u_n$ to basis $w_1, \ldots, w_n$:

$$u_j = p^i_j w_i.$$

Then

$$u_1 \wedge \cdots \wedge u_n = \det P \, w_1 \wedge \cdots \wedge w_n.$$

Definition. $u_1, \ldots, u_n$ has *same orientation as* $w_1, \ldots, w_n$ if $u_1 \wedge \cdots \wedge u_n$ is a positive multiple of $w_1 \wedge \cdots \wedge w_n$, i.e. $\det P > 0$. Otherwise *opposite orientation as*, i.e. $\det P < 0$.

'Same orientation as' is an equivalence relation on the set of all bases for M. There are just two equivalence classes. We call M an *oriented* vector space if one of these classes has been designated as *positively oriented bases* and the other as *negatively oriented bases*. We call this *choosing an orientation for* M.

For $\mathbb{R}^n$ we may designate the equivalence class of the usual basis $e_1, \ldots, e_n$ as being positively oriented bases. This is called the *usual orientation of* $\mathbb{R}^n$.

Example: In $\mathbb{R}^3$, with usual orientation. e_1, e_2, e_3 (see Figure 11.1) is positively oriented (by definition).

$$e_2 \wedge e_1 \wedge e_3 = -e_1 \wedge e_2 \wedge e_3,$$
$$e_2 \wedge e_3 \wedge e_1 = e_1 \wedge e_2 \wedge e_3.$$

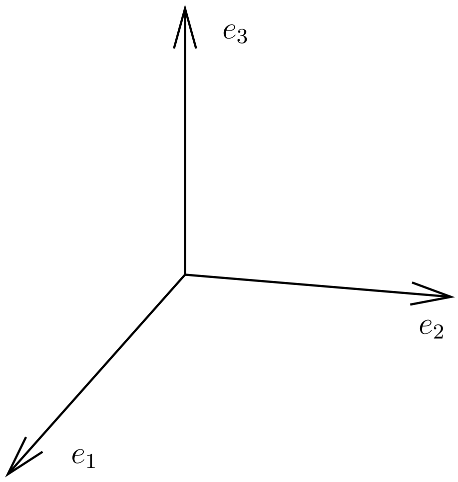

Figure 11.1

Therefore e_2, e_1, e_3 is negatively oriented and e_2, e_3, e_1 is positively oriented.

Definition. Let M be a real vector space of finite dimension n with a non-singular symmetric scalar product $(\cdot|\cdot)$. We call $u_1, \ldots, u_n$ a *standard basis* if

$$(u_i|u_j) = \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}.$$

Recall that such bases for M exist, and the numbers of $-$ signs is uniquely determined.

Theorem 11.1. *Let M be oriented. Then the n-form*

$$\mathrm{vol} = u^1 \wedge u^2 \wedge \cdots \wedge u^n$$

is independent of the choice of positively oriented standard basis for M. It is called the volume form *on M.*

If $v_1, \ldots, v_n$ is any positively oriented basis for M then

$$\mathrm{vol} = \sqrt{(-1)^s \det(v_i|v_j)}\, v^1 \wedge \cdots \wedge v^n.$$

Proof ▶

1. Let $w_1, \dots, w_n$ be another positively oriented standard basis for M: $w^i = p^i_j u^j$ (say). Therefore

$$w^1 \wedge \cdots \wedge w^n = \det P \, u^1 \wedge \cdots \wedge u^n.$$

But $\det P > 0$ and

$$P^t \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix} P = \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}.$$

Therefore

$$(-1)^s (\det P)^2 = (-1)^s.$$

Therefore

$$(\det P)^2 = 1.$$

Therefore $\det P = 1$. Therefore

$$w^1 \wedge \cdots \wedge w^n = u^1 \wedge \cdots \wedge u^n = \text{vol},$$

as required.

2. Let $u^i = p^i_j v^j$ (say), $\det P > 0$. Then

$$P^t \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix} P = G,$$

where $g_{ij} = (v_i | v_j)$. Therefore

$$(-1)^s (\det P)^2 = \det G.$$

Therefore

$$\det P = \sqrt{(-1)^s \det G}.$$

Therefore

$$\text{vol} = u^1 \wedge \cdots \wedge u^n = \det P \, v^1 \wedge \cdots \wedge v^n = \sqrt{(-1)^s \det G} \, v^1 \wedge \cdots \wedge v^n,$$

as required. ◄

Corollary 11.1. vol *has components* $\sqrt{|\det g_{ij}|}\, \epsilon_{i_1 \dots i_n}$ *wrt any positively oriented basis.*

Example: Take $\mathbb{R}^n$ with usual orientation and dot product. Let

$$D = \{t^1 v_1 + \cdots + t^n v_n : 0 \le t^i \le 1\}$$

be the parallelogram spanned by vectors $v_1, \ldots, v_n$. Let A be the matrix having $v_1, \ldots, v_n$ as columns.

$$\mathbb{R}^n \xrightarrow{A} \mathbb{R}^n, \quad v_i = Ae_i.$$

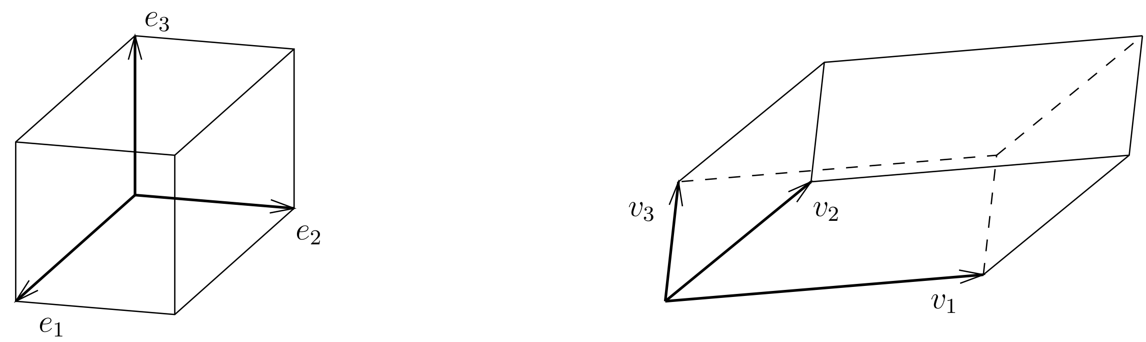

Figure 11.2

(for example, see Figure 11.2).

$$\begin{aligned}
\operatorname{vol}(v_1, \ldots, v_n) &= \operatorname{vol}(Ae_1, \ldots, Ae_n)\\
&= \det A \operatorname{vol}(e_1, \ldots, e_n)\\
&= \det A\\
&= \pm|\det A|\\
&= \pm\text{Lebesgue measure of } D,
\end{aligned}$$

and Lebesgue measure of $D = \sqrt{|\det(v_i|v_j)|}$.

We continue to consider a real oriented vector space M of finite dimension n with a non-singular symmetric scalar product $(\cdot|\cdot)$ with s signs.

$M^{(r)}$ denotes the vector space of skew-symmetric tensors of type

$$\underbrace{M \times \cdots \times M}_{\leftarrow r \rightarrow} \to K.$$

Theorem 11.2. *There exists a unique linear operator*

$$M^{(r)} \xrightarrow{*} M^{(n-r)},$$

called the Hodge star operator*, with the property that for each positively oriented standard basis $u_1, \dots, u_n$ we have*

$$*(u^1 \wedge \cdots \wedge u^r) = s_{r+1} \dots s_n u^{r+1} \wedge \cdots \wedge u^n \quad \textit{(no summation here)},$$

where

$$g_{ij} = \begin{pmatrix} s_1 & & 0 & \\ & s_2 & & \\ & & \ddots & \\ & 0 & & s_n \end{pmatrix} = g^{ij}, \quad s_i = \pm 1.$$

Example: If M is 3-dimensional oriented Euclidean, and u_1, u_2, u_3 is any positively oriented orthonormal then

$$M^{(1)} \xrightarrow{*} M^{(2)}, \quad M^{(2)} \to M^{(1)},$$

with

$$*(\alpha_1 u^1 + \alpha_2 u^2 + \alpha_3 u^3) = \alpha_1 \, u^2 \wedge u^3 + \alpha_2 \, u^3 \wedge u^1 + \alpha_3 \, u^1 \wedge u^2,$$
$$*(\alpha_1 u^2 \wedge u^3 + \alpha_2 u^3 \wedge u^1 + \alpha_3 u^1 \wedge u^2) = \alpha_1 u^1 + \alpha_2 u^2 + \alpha_3 u^3.$$

Thus, if v has components α_i wrt u^i, and w has components β_i wrt u^i then $*(v \wedge w)$ has components $\epsilon^{ijk}\alpha_j\beta_k$ wrt u^i for *any* positively oriented orthonormal basis u_i.

We write $v \times w = *(v \wedge w)$, and call it the *vector product* of v and w because:

$$\begin{aligned} v \times w &= *(v \wedge w) \\ &= *[(\alpha_1 u^1 + \alpha_2 u^2 + \alpha_3 u^3) \wedge (\beta_1 u^1 + \beta_2 u^2 + \beta_3 u^3)] \\ &= *[(\alpha_2\beta_3 - \alpha_3\beta_2) u^2 \wedge u^3 + \dots] \\ &= (\alpha_2\beta_3 - \alpha_3\beta_2) u^1 + \dots, \end{aligned}$$

as required.

Proof ►(of theorem) If $*$ exists then it must be unique, from the definition. Thus it is sufficient to define one such operator $*$. For any positively oriented basis we define $*\omega$ by contraction as:

$$(*\omega)_{i_{r+1} \dots i_n} = \frac{(-1)^s}{r!} g^{i_1 j_1} \dots g^{i_r j_r} \underbrace{\omega_{j_1 \dots j_r}}_{\omega} \underbrace{\sqrt{|\det g_{ij}|}\, \epsilon_{i_1 \dots i_r i_{r+1} \dots i_n}}_{\text{vol}}.$$

$*\omega$ is then well-defined independent of choice of basis, since contraction is independent of a choice of basis. Thus wrt a positively oriented standard basis $u_1, \dots, u_n$,

$$g^{ij} = \left\{ \begin{array}{cc} s_i & i = j \\ 0 & i \neq j \end{array} \right\} \text{ and } \omega = u^1 \wedge \dots \wedge u^r.$$

$\omega_{i\dots r} = 1$, other components of ω by skew-symmetry, otherwise zero. Therefore

$$(*\omega)_{r+1,\dots,n} = \frac{(-1)^s}{r!} s_1 s_2 \dots s_r \omega_{1\dots r} r! 1.1 = s_{r+1} \dots s_n,$$

as required as other components of $*\omega$ by skew-symmetry are zero.. ◄

Theorem 11.3. *There is a unique scalar product* $(\cdot|\cdot)$ *on each* $M^{(r)}$ *such that*

$$\omega \wedge \eta = (*\omega|\eta) \operatorname{vol}$$

for each $\omega \in M^{(r)}$, $\eta \in M^{(n-r)}$. *The scalar product is non-singular and symmetric for a standard basis*

$$(u^1 \wedge \dots \wedge u^r | u^1 \wedge \dots \wedge u^r) = (u^1|u^1) \dots (u^r|u^r) = \pm 1,$$

and $u^1 \wedge \dots u^r$ *is orthogonal to the other basis element of* $\{u^{i_1} \wedge \dots \wedge u^{i_r}\}_{i_1 < \dots < i_r}$.

Proof ► Define $(\cdot|\cdot)$ by $\omega \wedge \eta = (*\omega|\eta) \operatorname{vol}$. Then $(\cdot|\cdot)$ is a bilinear form on $M^{(n-r)}$.

If $u_1, \dots, u_n$ is a positively oriented standard basis for M then

$$\begin{aligned} \operatorname{vol} &= \underbrace{u^1 \wedge \dots \wedge u^r}_{\omega} \wedge \underbrace{u^{r+1} \wedge \dots \wedge u^n}_{\eta} \\ &= (\underbrace{s_{r+1} \dots s_n u^{r+1} \wedge \dots \wedge u^n}_{*\omega} \,|\, \underbrace{u^{r+1} \wedge \dots \wedge u^n}_{\eta}) \operatorname{vol}. \end{aligned}$$

Therefore

$$(u^{r+1} \wedge \dots \wedge u^n | u^{r+1} \wedge \dots \wedge u^n) = s_{r+1} \dots s_n = (u^{r+1}|u^{r+1}) \dots (u^n|u^n),$$

as required. Similarly other scalar products give zero. ◄

Similarly other scalar products give zero.

Example: If u_1, u_2, u_3 is an orthonormal basis for M then $u^2 \wedge u^3, u^3 \wedge u^1, u^1 \wedge u^3$ is an orthonormal basis for $M^{(2)}$, since

$$\begin{aligned} (u^2 \wedge u^3 | u^2 \wedge u^3) &= (u^2|u^2)(u^3|u^3) = 1.1 = 1, \\ (u^2 \wedge u^3 | u^3 \wedge u^1) &= (u^2|u^3)(u^3|u^1) = 0.0 = 0. \end{aligned}$$

11.2 Orientation of Coordinate Systems

Definition. Let X be an n-dimensional manifold. Two coordinate systems on X: $y^1, \ldots, y^n$ with domain V, and $z^1, \ldots, z^n$ with domain W have the *same orientation* if

$$\frac{\partial(y^1, \ldots, y^n)}{\partial(z^1, \ldots, z^n)} > 0$$

on $V \cap W$. We call X *oriented* if a family of coordinate systems is given on X whose domains cover X, and such that any two have the same orientation. We then call these coordinate systems *positively oriented*.

Note. On $V \cap W$,

$$dy^i = \frac{\partial y^i}{\partial z^j} dz^j.$$

Therefore

$$\begin{aligned} dy^1 \wedge \cdots \wedge dy^n &= \det \left(\frac{\partial y^i}{\partial z^j} \right) dz^1 \wedge \cdots \wedge dz^n \\ &= \frac{\partial(y^1, \ldots, y^n)}{\partial(z^1, \ldots, z^n)} dz^1 \wedge \cdots \wedge dz^n. \end{aligned}$$

Therefore for each $a \in V \cap W$, $\frac{\partial}{\partial y^1_a}, \ldots, \frac{\partial}{\partial y^n_a}$ has same orientation as $\frac{\partial}{\partial z^1_a}, \ldots, \frac{\partial}{\partial z^n_a}$. Thus each tangent space T_aX is an n-dimensional oriented vector space.

If X has a metric tensor $(\cdot|\cdot)$ then T_aX has a non-singular symmetric scalar product $(\cdot|\cdot)_a$ for each $a \in X$. Therefore we can define a differential n-form vol on X, called the *volume form* on X by:

$$(\mathrm{vol})_a = \text{ the volume form on } T_aX.$$

Also, if ω is a differential r-form on X then we can define a differential $(n-r)$-form on X, $*\omega$, called the *Hodge star* of ω by

$$(*\omega)_a = *(\omega_a) \quad \text{for each } a \in X.$$

If $u_1, \ldots, u_n$ are positively oriented vector fields on X with domain V, and

$$ds^2 = \pm(u^1)^2 \pm \cdots \pm (u^n)^2$$

then

$$\mathrm{vol} = u^1 \wedge \cdots \wedge u^n$$

on V.

If $y^1, \ldots, y^n$ is a positively oriented coordinate system on X with domain V then

$$\mathrm{vol} = \sqrt{|\det g_{ij}|}\, dy^1 \wedge \cdots \wedge dy^n$$

on V, where

$$g_{ij} = \left(\frac{\partial}{\partial y^i} \Big| \frac{\partial}{\partial y^j} \right).$$

Examples:

1. $\mathbb{R}^2$, with usual coordinates: x, y, and polar coordinates: r, θ (see Figure 11.3). Take x, y as positively oriented:

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Therefore

$$dx \wedge dy = (\cos\theta\, dr - r\sin\theta\, d\theta) \wedge (\sin\theta\, dr + r\cos\theta\, d\theta) = r\, dr \wedge d\theta.$$

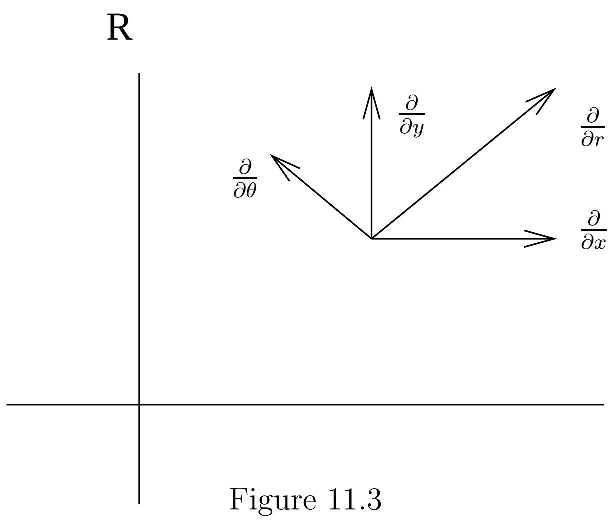

Figure 11.3

$r > 0$. Therefore r, θ is positively oriented.

$$\text{area element } = dx \wedge dy = r\, dr \wedge d\theta,$$
$$ds^2 = (dx)^2 + (dy)^2 = (dr)^2 + (r\, d\theta)^2.$$

Therefore

$$*dx = dy, \quad *\, dy = -dx,$$
$$*dr = r\, d\theta, \quad *\, (r\, d\theta) = -dr.$$

2. Unit sphere S^2 in $\mathbb{R}^3 : x^2 + y^2 + z^2 = 1$. On S^2 we have:

$$x\,dx + y\,dy + z\,dz = 0.$$

Therefore (wedge with dx):

$$y\,dx \wedge dy + z\,dx \wedge dz = 0.$$

Therefore

$$dx \wedge dy = \frac{z}{y} dz \wedge dx.$$

Therefore the coordinate system x, y on $z > 0$ has the same orientation as the coordinate system z, x on $y > 0$.

We orient S^2 so that these coordinates are positively oriented. Now

$$\begin{aligned}
ds^2 &= (dx)^2 + (dy)^2 + (dz)^2 \\
&= (dx)^2 + (dy)^2 + \left(-\frac{x}{z}dx - \frac{y}{z}dy\right)^2 \quad \text{(on } z > 0\text{)} \\
&= \left(1 + \frac{x^2}{z^2}\right)(dx)^2 + 2\frac{xy}{z^2}dx\,dy + \left(1 + \frac{y^2}{z^2}\right)(dy)^2.
\end{aligned}$$

Therefore wrt coordinates x, y,

$$g_{ij} = \begin{pmatrix} 1 + \frac{x^2}{z^2} & \frac{xy}{z^2} \\ \frac{xy}{z^2} & 1 + \frac{y^2}{z^2} \end{pmatrix}.$$

Therefore

$$\det g_{ij} = 1 + \frac{x^2}{z^2} + \frac{y^2}{z^2} = \frac{x^2 + y^2 + z^2}{z^2} = \frac{1}{z^2}.$$

Therefore

$$\text{area element } = \frac{1}{|z|} dx \wedge dy.$$

Chapter 12

Manifolds and (n)-dimensional Vector Analysis

This chapter could be considered a continuation of Chapter 11.

12.1 Gradient

Definition. If f is a scalar field on a manifold X, with non-singular metric tensor $(\cdot|\cdot)$, then we define the *gradient of* f to be the vector field $\operatorname{grad} f$ such that

$$(\operatorname{grad} f|v) =< df|v >= vf = \text{rate of change of } f \text{ along } v$$

for each vector field v.

Thus $\operatorname{grad} f$ is raising the index of df.

$$df = \frac{\partial f}{\partial y^i} dy^i$$

has components

$$\frac{\partial f}{\partial y^i},$$

and

$$\operatorname{grad} f = g^{ij} \frac{\partial f}{\partial y^j} \frac{\partial}{\partial y^i}$$

has components

$$g^{ij} \frac{\partial f}{\partial y^j}.$$

Theorem 12.1. *If the metric tensor is positive definite then* *(i)* grad f *is in the direction of fastest increase of* f*;*

(ii) the rate of change of f *in the direction of fastest increase is* $\|\operatorname{grad} f\|$*;*

(iii) grad f *is orthogonal to the level surfaces of* f *(see Figure 12.1).*

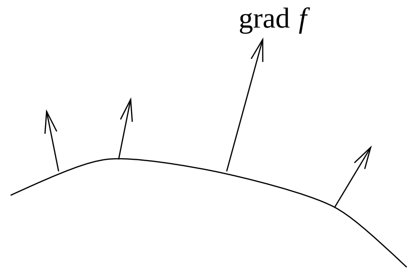

Figure 12.1

Proof ▶

(i) The rate of change of f along any unit vector field v has absolute value

$$|vf| = |(\operatorname{grad} f|v)| \leq \|\operatorname{grad} f\|\|v\| = \|\operatorname{grad} f\|,$$

by Cauchy-Schwarz.

(ii) For

$$v = \frac{\operatorname{grad} f}{\|\operatorname{grad} f\|}$$

the maximum is attained:

$$|vf| = \left|\left(\operatorname{grad} f|\frac{\operatorname{grad} f}{\|\operatorname{grad} f\|}\right)\right| = \|\operatorname{grad} f\|.$$

(iii) If v is tangented to the level surface $f = c$ then

$$(\operatorname{grad} f|v) = vf = 0.$$

◀

Definition. If f is a scalar field on a $2n$-dimensional manifold X, with coordinates

$$x^i = (q^1 \ldots q^n \; p_1 \ldots p_n)$$

and skew-symmetric tensor

$$(\cdot|\cdot) = \sum_{i=1}^{n} dp_i \wedge dq^i$$

then along a curve α whose velocity vector is $\operatorname{grad} f$ we have:

$$\frac{dx^i}{dt} = g^{ij}\frac{\partial}{\partial x^j},$$

i.e.

$$\begin{pmatrix} \frac{dq^1}{dt} \\ \vdots \\ \frac{dq^n}{dt} \\ \frac{dp_1}{dt} \\ \vdots \\ \frac{dp_n}{dt} \end{pmatrix} = \left(\begin{array}{c|c} 0 & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} \\ \hline \begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix} & 0 \end{array}\right) \begin{pmatrix} \frac{\partial f}{\partial q^1} \\ \vdots \\ \frac{\partial f}{\partial q^n} \\ \frac{\partial f}{\partial p_1} \\ \vdots \\ \frac{\partial f}{\partial p_n} \end{pmatrix},$$

i.e.

$$\begin{aligned} \frac{dq^i}{dt} &= \frac{\partial f}{\partial p_i}, \\ \frac{dp_i}{dt} &= -\frac{\partial f}{\partial q^i} \end{aligned}$$

(*Hamiltonian Equations of Motion*).

Note.

$$\frac{d}{dt}f(\alpha(t)) = \frac{\partial f}{\partial x^i}(\alpha(t))\frac{dx^i}{dt}(\alpha(t)) = g^{ij}(\alpha(t))\frac{\partial f}{\partial x^j}(\alpha(t))\frac{\partial f}{\partial x^i}(\alpha(t)) = 0,$$

since g^{ij} is skew-symmetric.

i.e.

$$\text{rate of change of } f \text{ along } \operatorname{grad} f \ = \langle df, \operatorname{grad} f\rangle = (\operatorname{grad} f|\operatorname{grad} f) = 0,$$

since $(\cdot|\cdot)$ is skew-symmetric.

12.2 3-dimensional Vector Analysis

Given X a 3-dimensional manifold, x, y, z positively oriented orthonormal coordinates, i.e. metric tensor with line element $(dx)^2 + (dy)^2 + (dz)^2$. Write

$$\nabla = \left(\frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\ \frac{\partial}{\partial z}\right), \qquad dr = (dx\ dy\ dz),$$

$$dS = (dy \wedge dz,\ dz \wedge dx,\ dx \wedge dy), \qquad dV = dx \wedge dy \wedge dz,$$

$$F = (F^1\ F^2\ F^3) = (F_1\ F_2\ F_3), \qquad \nabla f = \left(\frac{\partial f}{\partial x}\ \frac{\partial f}{\partial y}\ \frac{\partial f}{\partial z}\right) \quad \text{for a scalar field } f,$$

$$\nabla \times F = \left(\frac{\partial F^3}{\partial y} - \frac{\partial F^2}{\partial z},\quad ,\quad \right), \qquad \nabla.F = \frac{\partial F^1}{\partial x} + \frac{\partial F^2}{\partial y} + \frac{\partial F^3}{\partial z}.$$

The vector field

$$\vec{F} = F.\nabla = F^1\frac{\partial}{\partial x} + F^2\frac{\partial}{\partial y} + F^3\frac{\partial}{\partial z}$$

components F, corresponds, lowering the index, to the 1-form (components F)

$$\begin{aligned} F.dr &= F_1 dx + F_2 dy + F_3 dz, \\ df &= (\nabla f).dr, \\ d[F.dr] &= (\nabla \times F).dS, \\ d[F.dS] &= (\nabla.F)dV, \\ *1 &= dV, \\ *dr &= dS, \\ *dS &= dr, \\ *V &= 1. \end{aligned}$$

Now

$$\begin{array}{ccccccc} & & \text{1-forms} & & \text{2-forms} & & \text{3-froms} \\ \Omega(X) & \to & \Omega^1(X) & \to & \Omega^2(X) & \to & \Omega^3(X) \\ & & F.dr & \overset{d}{\mapsto} & (\nabla \times F).dS & & \\ & & & \searrow * & & & \\ f & \mapsto & \nabla F.dr & & F.dS & \overset{d}{\mapsto} & (\nabla.F)dV \\ & & (\nabla \times F).dr, & & & & \\ & & \vec{F}, & & & & \\ & & \text{grad}\, f, & & & & \\ & & \text{curl}\, \vec{F} & & & & \end{array}$$

12.3 Results

Field	Components	Type
grad f	∇f	vector
curl $\vec{F}$	$\nabla \times F$	vector
div $\vec{F}$	$\nabla . F$	scalar

Theorem 12.2. *Let v be a vector field on a manifold, with non-singular symmetric metric tensor. Let ω be the 1-form given by lowering the index of v. Then the scalar field div$v$ defined by:*

$$d * \omega = (\operatorname{div} v)\,\mathrm{vol}$$

is called the divergence of v. *If v has components v^i wrt coordinates y^i then*

$$\operatorname{div} v = \frac{1}{\sqrt{g}} \frac{\partial}{\partial y^i}(\sqrt{g}\, v^i),$$

where $g = |\det g_{ij}|$.

Proof ▶

$$(*\omega)_{j_1 \dots j_{n-1}} = g^{ij} v_j \sqrt{g}\, \epsilon_{ij_1 \dots j_{n-1}} = \sqrt{g}\, v^i \epsilon_{ij_1 \dots j_{n-1}}.$$

Therefore

$$*\omega = \sqrt{g}\, v^1 dy^2 \wedge dy^3 \wedge \dots \wedge dy^n - \sqrt{g}\, v^2 dy^1 \wedge dy^3 \wedge \dots \wedge dy^n + \dots .$$

Therefore

$$d * \omega = \frac{\partial}{\partial y^i}(\sqrt{g}\, v^i) dy^1 \wedge dy^2 \wedge \dots \wedge dy^n = \frac{1}{\sqrt{g}} \frac{\partial}{\partial y^i}(\sqrt{g}\, v^i)\,\mathrm{vol},$$

as required. ◀

Theorem 12.3. *Let f be a scalar field on a manifold, with non-singular symmetric metric tensor. Then the scalar field*

$$\Delta f = \operatorname{div} \operatorname{grad} f = *d * df$$

is called the Laplacian of f. *Wrt coordinates y^i we have*

$$\Delta f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial y^i} \left(\sqrt{g}\, g^{ij} \frac{\partial f}{\partial y^j} \right).$$

Thus

$$\Delta f = \frac{1}{\sqrt{g}} \begin{pmatrix} \frac{\partial}{\partial y^1} & \dots & \frac{\partial}{\partial y^n} \end{pmatrix} \sqrt{g} \begin{pmatrix} g^{11} & \dots & g^{1n} \\ \vdots & & \vdots \\ g^{n1} & \dots & g^{nn} \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial y^1} \\ \vdots \\ \frac{\partial f}{\partial y^n} \end{pmatrix}.$$

Examples:

(i) $\mathbb{R}^n$, with $(ds)^2 = (dx^1)^2 + \cdots + (dx^n)^2$, $(g^{ij}) = I$, $g = 1$. Therefore

$$\Delta f = \begin{pmatrix} \frac{\partial}{\partial x^1} & \cdots & \frac{\partial}{\partial x^n} \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x^1} \\ \vdots \\ \frac{\partial}{\partial x^n} \end{pmatrix} = \frac{\partial^2 f}{\partial {x^1}^2} + \cdots + \frac{\partial^2 f}{\partial {x^n}^2}.$$

Therefore

$$\Delta = \frac{\partial^2}{\partial {x^1}^2} + \cdots + \frac{\partial^2}{\partial {x^n}^2} \quad \textit{usual Laplacian.}$$

(ii) $\mathbb{R}^4$, with $(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 - (dt)^2$: Minkowski.

$$\Delta f = \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial t} \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \\ \frac{\partial f}{\partial t} \end{pmatrix}$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} - \frac{\partial^2 f}{\partial t^2}.$$

Therefore

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2} \quad \textit{wave operator}.$$

(iii) S^2, with $(ds)^2 = (d\theta)^2 + (\sin\theta\, d\varphi)^2$, $g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$, $g = \sin^2\theta$.

$$\begin{aligned} \Delta f &= \frac{1}{\sin\theta} \left(\frac{\partial}{\partial\theta} \; \frac{\partial}{\partial\varphi} \right) \sin\theta \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial\theta} \\ \frac{\partial f}{\partial\varphi} \end{pmatrix} \\ &= \frac{1}{\sin\theta} \left(\frac{\partial}{\partial\theta} \; \frac{\partial}{\partial\varphi} \right) \begin{pmatrix} \sin\theta \frac{\partial f}{\partial\theta} \\ \frac{1}{\sin\theta}\frac{\partial f}{\partial\varphi} \end{pmatrix} \\ &= \frac{1}{\sin\theta} \left[\frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial f}{\partial\theta} \right) + \frac{\partial}{\partial\varphi} \left(\frac{1}{\sin\theta} \frac{\partial f}{\partial\varphi} \right) \right]. \end{aligned}$$

Definition. Let X be a 3-dimensional oriented manifold with non-singular symmetric metric tensor. Let v be a vector field corresponding to the 1-form ω. Then $\operatorname{curl} v$ is the vector field corresponding to the 1-form $*d\omega$. Wrt positively oriented coordinates y^i, $\operatorname{curl} v$ has components:

$$\epsilon^{ijk} \frac{1}{\sqrt{g}} \left(\frac{\partial v_k}{\partial y^j} - \frac{\partial v_j}{\partial y^k} \right),$$

where $g = |\det g_{ij}|$.

12.4 Closed and Exact Forms

Definition. A differential form $\omega \in \Omega^r(X)$ is called

(i) *closed* if $d\omega = 0$;

(ii) *exact* if $\omega = d\eta$ for some $\eta \in \Omega^{r-1}(X)$.

We note that

1. ω exact $\Rightarrow$ ω closed $(d\,d\eta = 0)$,
2. ω an exact 1-form $\Rightarrow \omega = df$ (say)

$$\Rightarrow \int_\alpha \omega = \int_\alpha df = 0$$

for each closed curve α.

Examples:

1. If $\omega = P\,dx + Q\,dy$ is a 1-form on an open $V \subset \mathbb{R}^2$ then

$$\begin{aligned}
\omega &= dP \wedge dx + dQ \wedge dy \\
&= \left(\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy\right) \wedge dx + \left(\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy\right) \wedge dy \\
&= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy.
\end{aligned}$$

Therefore

$$\omega \text{ is closed} \Leftrightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \text{ on } V,$$
$$\omega \text{ is exact} \Leftrightarrow P = \frac{\partial f}{\partial x}, Q = \frac{\partial f}{\partial y}$$

for some scalar field f on V.

2. The 1-form

$$\omega = \frac{x\,dy - y\,dx}{x^2 + y^2}$$

on $\mathbb{R}^2 - \{0\}$ is called the *angle-form* about 0. We have:

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = +\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2},$$
$$\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = -\frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}.$$

Therefore ω is closed.

ω is not exact because if $\alpha(t) = (\cos t, \sin t)$ is the unit circle about 0 $(0 \le t \le 2\pi)$ then

$$\int_\alpha \omega = \int_0^{2\pi} \frac{\cos t.\sin t - \sin t.(-\sin t)}{\cos^2 t + \sin^2 t} dt = \int_0^{2\pi} 1\, dt = 2\pi \neq 0.$$

However, on $\mathbb{R}^2 - \{\text{negative or zero } x\text{-axis}\}$ we have

$$x = r\cos\theta, \quad y = r\sin\theta,$$

where θ is a scalar field, with $-\pi \le \theta \le \pi$ and

$$\omega = \frac{r\cos\theta.(-r\cos\theta) - r\sin\theta.(-r\sin\theta)}{r^2\cos^2\theta + r^2\sin^2\theta} d\theta = d\theta.$$

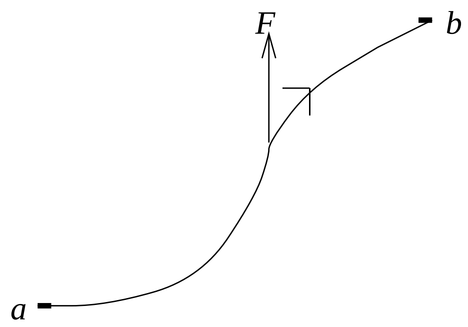

Figure 12.2

Therefore, if α is a curve from a to b (see Figure 12.2),

$$\int_\alpha \omega = \int_\alpha d\theta = \theta(b) - \theta(a) = \text{ change in angle along } \alpha.$$

Note.

$$\frac{dz}{z} = \frac{\overline{z}\, dz}{\overline{z}z} = \frac{(x - iy)(dx + i\, dy)}{x^2 + y^2} = \frac{x\, dx + y\, dy}{x^2 + y^2} + i\frac{x\, dy - y\, dx}{x^2 + y^2}.$$

Therefore $\omega = \text{im}\, \frac{\partial z}{\partial}$.

3. If

$$\vec{F} = F^1 \frac{\partial}{\partial x} + F^2 \frac{\partial}{\partial y} + F^3 \frac{\partial}{\partial z}$$

is a force field in $\mathbb{R}^3$ then

$$F.dr = F_1 dx + F_2 dy + F_3 dz, \quad F_i = F^i$$

is called the *work element.*

$$\int_\alpha F.dr = \textit{work done} \text{ by the force } \vec{F} \text{ along the curve } \alpha.$$

$\vec{F}$ is *conservative* if $F.dr$ is exact, i.e.

$$F.dr = dV,$$

where V is a scalar field. V is called a *potential function* for $\vec{F}$.

Work done by $\vec{F}$ along α from a to b (see Figure 12.3) is

$$\int_\alpha F.dr = \int_\alpha dV = V(b) - V(a) = \textit{potential difference}.$$

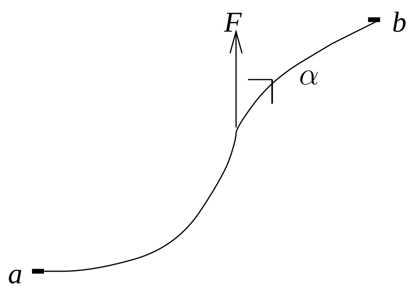

Figure 12.3

A necessary condition that $\vec{F}$ be conservative is that $F.dr$ be closed, i.e.

$$\nabla \times F = 0.$$

12.5 Contraction and Results

Definition. An open set $V \subset \mathbb{R}^n$ is called *contractible* to $a \in V$ if there exists a C^∞ map

$$V \times [0,1] \xrightarrow{\varphi} V$$

such that

$$\varphi(x,1) = x,$$
$$\varphi(x,0) = a$$

for all $x \in V$.

Example: V star-shaped $\Rightarrow$ V contractible. $\varphi(x,t) = tx + (1-t)a$ (see Figure 12.4).

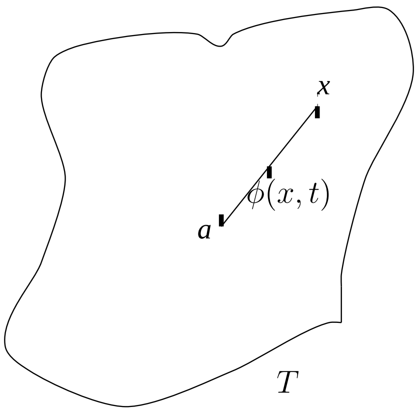

Figure 12.4

Theorem 12.4 (Poincaré Lemma). *Let $\omega \in \Omega^r(V)$, where V is a contractible open subset of $\mathbb{R}^n$, and $r \geq 1$. Then ω is exact iff ω is closed.*

Proof ▶ Let $I = [0,1]$ be the unit interval $0 \leq t \leq 1$, and define a linear operator ('homotopy')

$$\Omega^r(V \times I) \xrightarrow{H} \Omega^{r-1}(V)$$

for each $r \geq 1$ by

$$H[f\,dt \wedge dx^J] = \left(\int_0^1 f\,dt\right) dx^J,$$
$$H[f\,dx^J] = 0.$$

Now calculate the operator $dH + Hd$:

(i) if $\eta = f\,dt \wedge dx^J$ then

$$\begin{aligned} dH\eta + H\,d\eta &= d\left[\left(\int_0^1 f\,dt\right) dx^J\right] + H\left[-\frac{\partial f}{\partial x^i} dt \wedge dx^i \wedge dx^J\right] \\ &= \left(\int_0^1 \frac{\partial f}{\partial x^i} dt\right) dx^i \wedge dx^J - \left(\int_0^1 \frac{\partial f}{\partial x^i} dt\right) dx^i \wedge dx^J \\ &= 0; \end{aligned}$$

(ii) if $\eta = f\,dx^J$ then

$$\begin{aligned} dH\eta + H\,d\eta &= 0 + H\left[\frac{\partial f}{\partial t}dt \wedge dx^J + \frac{\partial f}{\partial x^i}dx^i \wedge dx^J\right] \\ &= \left(\int_0^1 \frac{\partial f}{\partial t}dt\right) dx^J + 0 \\ &= [f(x,t)]_{t=0}^{t=1} dx^J. \end{aligned}$$

(iii) Now let V be contractible, with

$$V \times I \xrightarrow{\varphi} V$$

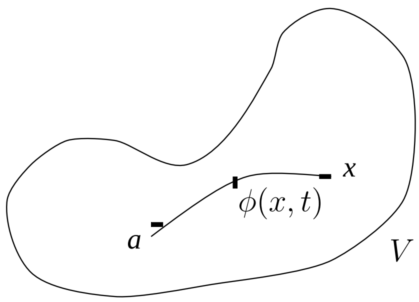

Figure 12.5

a C^∞ map such that (see Figure 12.5)

$$\begin{aligned} \varphi(x,1) &= x, \\ \varphi(x,0) &= a. \end{aligned}$$

So

$$\begin{aligned} \varphi^i(x,1) &= x^i, \\ \varphi^i(x,0) &= a^i. \end{aligned}$$

Therefore

$$\frac{\partial \varphi^i}{\partial x^i} = \begin{cases} \delta^i_j & \text{at } t = 1; \\ 0 & \text{at } t = 0. \end{cases}$$

Let $\omega \in \Omega^r(V)$, say $\omega = g(x)dx^{i_1} \wedge \cdots \wedge dx^{i_r}$. Apply φ^*:

$$\begin{aligned} \varphi^*\omega &= g(\varphi(x,t))d\varphi^{i_1} \wedge \cdots \wedge d\varphi^{i_r} \\ &= g(\varphi(x,t))\left[\frac{\partial \varphi^{i_1}}{\partial x^{j_1}}dx^{j_1} + \frac{\partial \varphi^{i_1}}{\partial t}dt\right] \wedge \cdots \wedge \left[\frac{\partial \varphi^{i_r}}{\partial x^{j_r}}dx^{j_r} + \frac{\partial \varphi^{i_r}}{\partial t}dt\right] \\ &= g(\varphi(x,t))\left[\frac{\partial \varphi^{i_1}}{\partial x^{j_1}} \cdots \frac{\partial \varphi^{i_r}}{\partial x^{j_r}}dx^{j_1} \wedge \cdots \wedge dx^{j_r} + \left(\frac{\partial \varphi^{i_1}}{\partial t} \cdots \frac{\partial \varphi^{i_r}}{\partial t}dt \wedge \cdots \wedge dt\right) \wedge dt\right]. \end{aligned}$$

Apply $dH + Hd$:

$$\begin{aligned}(dH + Hd)\varphi^*\omega &= \left[g(\varphi(x,t))\frac{\partial\varphi^{i_1}}{\partial x^{j_1}}\cdots\frac{\partial\varphi^{i_r}}{\partial x^{j_r}}\right]_{t=0}^{t=1} dx^{j_1}\wedge\cdots\wedge dx^{j_r} + 0\\ &= g(x)\delta^{i_1}_{j_1}\ldots\delta^{i_r}_{j_r}dx^{j_1}\wedge\cdots\wedge dx^{j_r}\\ &= g(x)dx^{i_1}\wedge\cdots\wedge dx^{i_r}\\ &= \omega.\end{aligned}$$

Hence:

$$(dH + Hd)\varphi^*\omega = \omega$$

for all $\omega \in \Omega^r(V)$.

(iv) Let ω be closed. Then

$$d\varphi^*\omega = \varphi^*d\omega = 0.$$

Therefore

$$dH\varphi^*\omega = \omega.$$

Therefore ω is exact. ◄

Theorem 12.5. *Let ω be a closed r-form, with domain V open in manifold X. Let $a \in V$. Then there exists an open neighbourhood W of a such that ω is exact on W.*

Proof ► Let y be a coordinate system on X at a, domain $U \subset V$, say. Let $W \subset U$ be an open neighbourhood of a such that $y(W)$ is an open ball (see Figure 12.8).

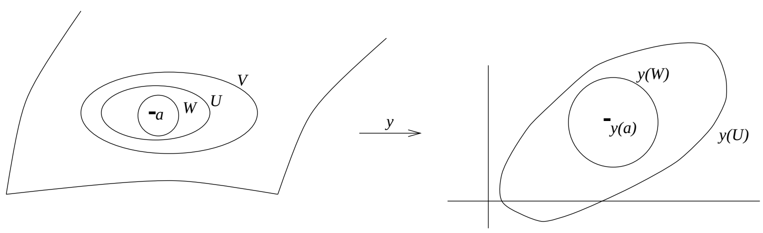

Figure 12.6

Consider

$$W \xrightarrow{y} y(W), \quad W \underset{\varphi}{\leftarrow} y(W)$$

(open ball), where φ is the inverse map.

$d\varphi^*\omega = \varphi^* d\omega = 0$, since ω is closed. Therefore $\varphi^*\omega$ is closed. Therefore $\varphi^*\omega = d\eta$ on $y(W)$, by Poincaré. Therefore

$$dy^*\eta = y^* d\eta = y^*\varphi^*\omega = \omega$$

on W. Therefore ω is exact on W. ◀

Theorem 12.6. *Let X be a 2-dimensional oriented manifold with a positive definite metric tensor. Let u_1, u_2 be positively oriented orthonormal vector fields on X, with domain V* (moving frame). *Then there exists a unique 1-form ω on V such that*

$$(*) \quad \begin{pmatrix} du^1 \\ du^2 \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \wedge \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}$$

on V. ω is called the connection form (gauge field) *wrt moving frame u_1, u_2.*

Proof ▶ Any 1-form ω on V can be written uniquely as:

$$\omega = \alpha u^1 + \beta u^2, \quad \alpha, \beta \text{ scalar fields.}$$

$$\begin{aligned} \omega \text{ satisfies } (*) \Leftrightarrow\ & du^1 = -\omega \wedge u^2, \\ & du^2 = \omega \wedge u^1 \\ \Leftrightarrow\ & du^1 = -(\alpha u^1 + \beta u^2) \wedge u^2 = -\alpha u^1 \wedge u^2, \\ & du^2 = (\alpha u^1 + \beta u^2) \wedge u^1 = -\beta u^1 \wedge u^2. \end{aligned}$$

Thus α, β are uniquely determined. ◀

Theorem 12.7. *Let X be a 2-dimensional oriented manifold with positive definite metric tensor. Let u_1, u_2 be a moving frame with domain V, with connection form ω and*

$$d\omega = K u^1 \wedge u^2 = K.\text{area element.}$$

Then the scalar field K is independent of the choice of moving frame, and is called the Gaussian curvature *of X.*

Proof ► Let w_1, w_2 be another moving frame with domain V:

$$\begin{pmatrix} w^1 \\ w^2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}$$

(say). Write this in matrix form as:

$$w = Pu.$$

Also

$$\begin{pmatrix} du^1 \\ du^2 \end{pmatrix} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \wedge \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}.$$

Write this in matrix form as:

$$du = \Omega \wedge u.$$

Then

$$\begin{aligned} dw &= (dP) \wedge u + P\,du \\ &= (dP) \wedge u + P\Omega \wedge u \\ &= (dP + P\Omega) \wedge u \\ &= (dP + P\Omega) \wedge P^{-1}w \\ &= [(dP)P^{-1} + P\Omega P^{-1}] \wedge w \\ &= \left[\begin{pmatrix} -\sin\theta\,d\theta & -\cos\theta\,d\theta \\ \cos\theta\,d\theta & -\sin\theta\,d\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right. \\ &\quad \left. + \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right] \wedge w \\ &= \left[\begin{pmatrix} 0 & -d\theta \\ d\theta & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}\right] \wedge w \\ &= \begin{pmatrix} 0 & -(\omega + d\theta) \\ \omega + d\theta & 0 \end{pmatrix} \wedge w. \end{aligned}$$

Therefore $\omega + d\theta$ is the connection form wrt moving frame w_1, w_2 and

$$d[\omega + d\theta] = d\omega + d\,d\theta = d\omega,$$

as required. ◄

Example: On S^2, with angle coordinates θ, φ,

$$ds^2 = (d\theta)^2 + (\sin\theta\,d\varphi)^2, \quad u^1 = d\theta, \quad u^2 = \sin\theta\,d\varphi.$$

Recall

$$\omega = -\cos\theta \, d\varphi,$$
$$d\omega = \sin\theta \, d\theta \wedge d\varphi = u^1 \wedge u^2.$$

Therefore Gaussian curvature is constant function 1.

Theorem 12.8. *Let X be an oriented 2-dimensional manifold with positive definite metric tensor. Then the Gaussian curvature of X is zero iff for each $a \in X$ there exists local coordinates x, y such that*

$$ds^2 = (dx)^2 + (dy)^2,$$

i.e.

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof ► Let u_1, u_2 be a moving frame with connection form ω on an open neighbourhood V of a on which the Poincaré lemma holds. Then, on V:

$$\begin{aligned}
\text{Gaussian curvature is zero} &\Leftrightarrow d\omega = 0 \\
&\Leftrightarrow \omega = -d\theta \quad \text{(say), by Poincaré} \\
&\Leftrightarrow \omega + d\theta = 0 \\
&\Leftrightarrow u_1, u_2 \text{ can be rotated to a new frame } w_1, w_2 \\
&\qquad \text{having connection form } 0 \\
&\Leftrightarrow dw^1 = 0, dw^2 = 0 \\
&\Leftrightarrow w^1 = dx, w^2 = dy \quad \text{(say), by Poincaré} \\
&\Leftrightarrow ds^2 = (dx)^2 + (dy)^2.
\end{aligned}$$

$\vec{\mathbf{N}}$ ◄

12.6 Surface in $\mathbb{R}^3$

Let X be a 2-dimensional submanifold of $\mathbb{R}^3$. Denote all vectors by their usual components in $\mathbb{R}^3$. Let $\vec{\mathbf{N}}$ be a field of unit vectors normal to X, t^1, t^2 be coordinates on X, and let $\vec{\mathbf{r}} = (x, y, z)$. Let $\frac{\partial \vec{\mathbf{r}}}{\partial t^1}, \frac{\partial \vec{\mathbf{r}}}{\partial t^2}$ be a basis for vectors tangent to X (see Figure 12.9).

$$\frac{\partial \vec{\mathbf{r}}}{\partial t^i} . \vec{\mathbf{N}} = 0.$$

Therefore

$$\frac{\partial^2 \vec{\mathbf{r}}}{\partial t^i \partial t^j}.\vec{\mathbf{N}} + \frac{\partial \vec{\mathbf{r}}}{\partial t^i}.\frac{\partial \vec{\mathbf{N}}}{\partial t^j} = 0.$$

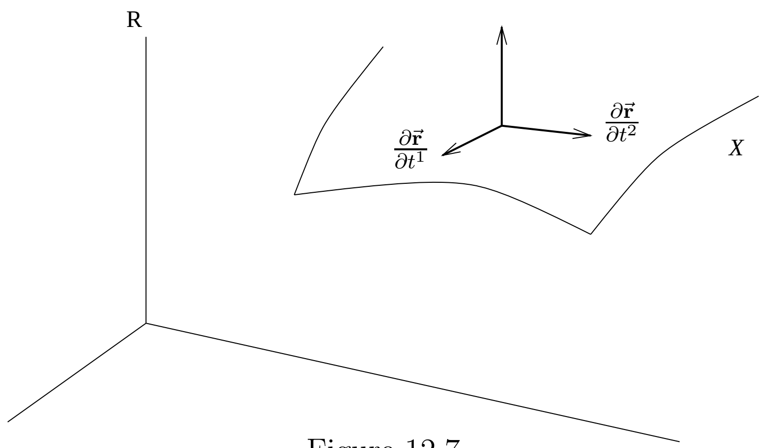

Figure 12.7

If $\vec{\mathbf{u}}$ is a vector field on X tangent to X then

$$\vec{\mathbf{N}}.\vec{\mathbf{N}} = 1$$

along $\vec{\mathbf{u}}$. Therefore

$$(\nabla_{\vec{\mathbf{u}}}\vec{\mathbf{N}}).\vec{\mathbf{N}} + \vec{\mathbf{N}}.\nabla_{\vec{\mathbf{u}}}\vec{\mathbf{N}} = 0.$$

Therefore $\nabla_{\vec{\mathbf{u}}}\vec{\mathbf{N}}$ is tangential to X.

So define tensor field S on X

$$S\vec{\mathbf{u}} = -\nabla_{\vec{\mathbf{u}}}\vec{\mathbf{N}}.$$

S is called the *shape operator*, and measures the amount of curvature of X in $\mathbb{R}^3$. Wrt coordinates t^1, t^2 S has covariant components

$$S_{ij} = \left(\frac{\partial}{\partial t^i}|S\frac{\partial}{\partial t^j}\right) = \frac{\partial \vec{\mathbf{r}}}{\partial t^i}. - \frac{\partial \vec{\mathbf{N}}}{\partial t^j} = \frac{\partial^2 x}{\partial t^i \partial t^j}.\vec{\mathbf{N}} \quad \text{(symmetric)}.$$

Therefore S_a is a self-adjoint operator on T_aX for each a. Therefore it has real eigenvalues K_1, K_2 and orthonormal eigenvectors u_1, u_2 (see Figure 12.8) exist, (say) $K_1 \geq K_2$.

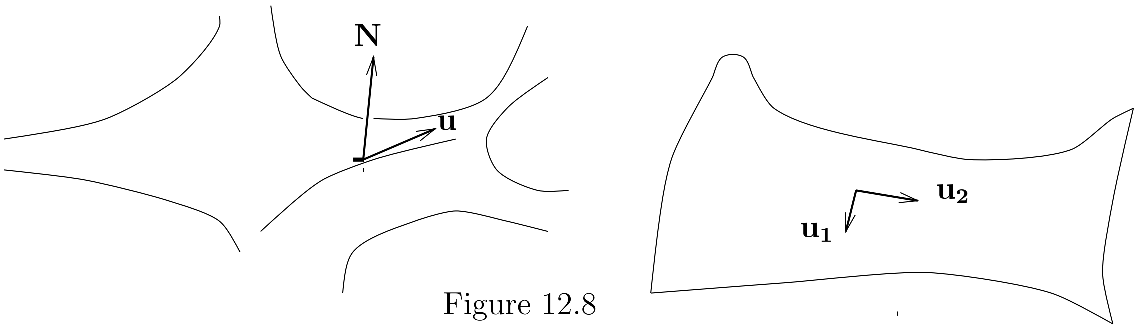

Figure 12.8

If we intersect X by a plane normal to X containing the vector

$$\cos\theta\, u_1 + \sin\theta\, u_2$$

at a, we have a curve α of intersection along which the unit tangent vector $\underline{t}$ satisfies:

$$\vec{\mathbf{t}}.\vec{\mathbf{N}} = 0.$$

Therefore

$$(\nabla_{\vec{\mathbf{t}}}\vec{\mathbf{t}}).\vec{\mathbf{N}} + \vec{\mathbf{t}}.(\nabla_{\vec{\mathbf{t}}}\vec{\mathbf{N}}) = 0.$$

Therefore

$$\kappa\vec{\mathbf{N}}.\vec{\mathbf{N}} - \vec{\mathbf{t}}.S\vec{\mathbf{t}} = 0$$

at a, where κ is the curvature of α at a. Therefore

$$\begin{aligned}\kappa = \vec{\mathbf{t}}.S\vec{\mathbf{t}} &= (\cos\theta\, u_1 + \sin\theta\, u_2).S(\cos\theta\, u_1 + \sin\theta\, u_2) \quad \text{(at } a\text{)}\\ &= \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta.\end{aligned}$$

Therefore u_1 is the direction of maximum curvature K_1, and u_2 is the direction of minimum curvature K_2.

Put $\vec{\mathbf{N}} = u_3$, with u_1, u_2, u_3 a moving frame.

$$\nabla u_3 = -\omega_3^1 \otimes u_1 - \omega_3^2 \otimes u_2 - \omega_3^3 \otimes u_3.$$

Therefore

$$\begin{aligned}Su_1 &= -\nabla_{u_1} u_3 = \langle \omega_3^1, u_1\rangle u_1 + \langle \omega_3^2, u_1\rangle u_2,\\ Su_2 &= -\nabla_{u_2} u_3 = \langle \omega_3^1, u_2\rangle u_1 + \langle \omega_3^2, u_2\rangle u_2.\end{aligned}$$

Therefore

$$\begin{aligned}\kappa_1\kappa_2 &= \det S\\ &= \langle \omega_3^1, u_1\rangle\langle \omega_3^2, u_2\rangle - \langle \omega_3^1, u_2\rangle\langle \omega_3^2, u_1\rangle\\ &= \omega_3^1 \wedge \omega_3^2(u_1, u_2)\\ &= d\omega_2^1(u_1, u_2)\\ &= \kappa\, u^1 \wedge u^2(u_1, u_2)\\ &= \kappa\end{aligned}$$

(since $d\Omega = -\Omega \wedge \Omega$, so $d\omega_2^1 = -\omega_k^1 \wedge \omega_2^k = -\omega_3^1 \wedge \omega_2^3 = \omega_3^1 \wedge \omega_3^2$).

12.7 Integration on a Manifold (Sketch)

Let ω be a differential n-form on an oriented n-dimensional manifold X. We want to define

$$\int_X \omega,$$

the *integral of* ω *over* X.

To justify in detail the construction which follows X must satisfy some conditions. It is sufficient, for instance, that X be a submanifold of some $\mathbb{R}^N$.

(i) Suppose

$$\begin{aligned}\omega &= f(y^1, \ldots, y^n)dy^1 \wedge \cdots \wedge dy^n\\ &= y^*[f(x^1, \ldots, x^n)dx^1 \wedge \cdots \wedge dx^n]\\ &= y^*\omega_1\end{aligned}$$

on the domain V of a positively oriented coordinate system y^i (see Figure 12.9), and that ω is zero outside V, and that

$$\operatorname{supp} f = \text{ closure of } \{x \in y(V) : f(x) \neq 0\}$$

is a bounded set contained in $y(V)$. Then we define

$$\int_X \omega = \int_{y(V)} \omega_1,$$

i.e.

$$\int_X f(y^1, \ldots, y^n) dy^1 \wedge \cdots \wedge dy^n = \int_{y(V)} f(x_1, \ldots, x_n) dx_1 \ldots dx_n$$

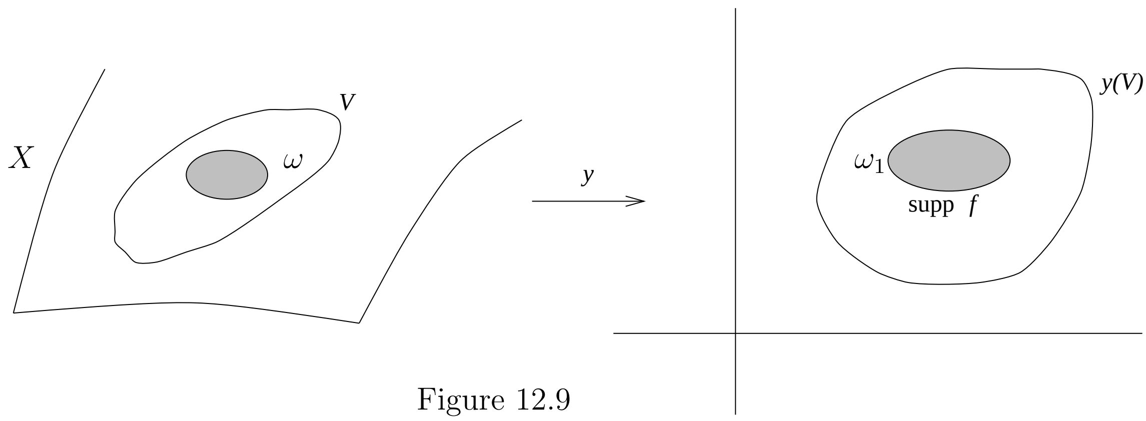

Figure 12.9

(Lebesgue integral). The definition of $\int_X \omega$ does not depend on the choice of coordinates, since if z^i with domain W is another such coordinate system,

$$\omega = z^* \omega_2$$

(say), then

$$\varphi^* \omega_2 = \omega_1,$$

where $\varphi = z \circ y^{-1}$ (see Figure 12.10). Therefore

$$\int_{y(V)} \omega_1 = \int_{y(V \cap W)} \omega_1 = \int_{y(V \cap W)} \varphi^* \omega_2 = \int_{z(V \cap W)} \omega_2 = \int_{z(W)} \omega_2.$$

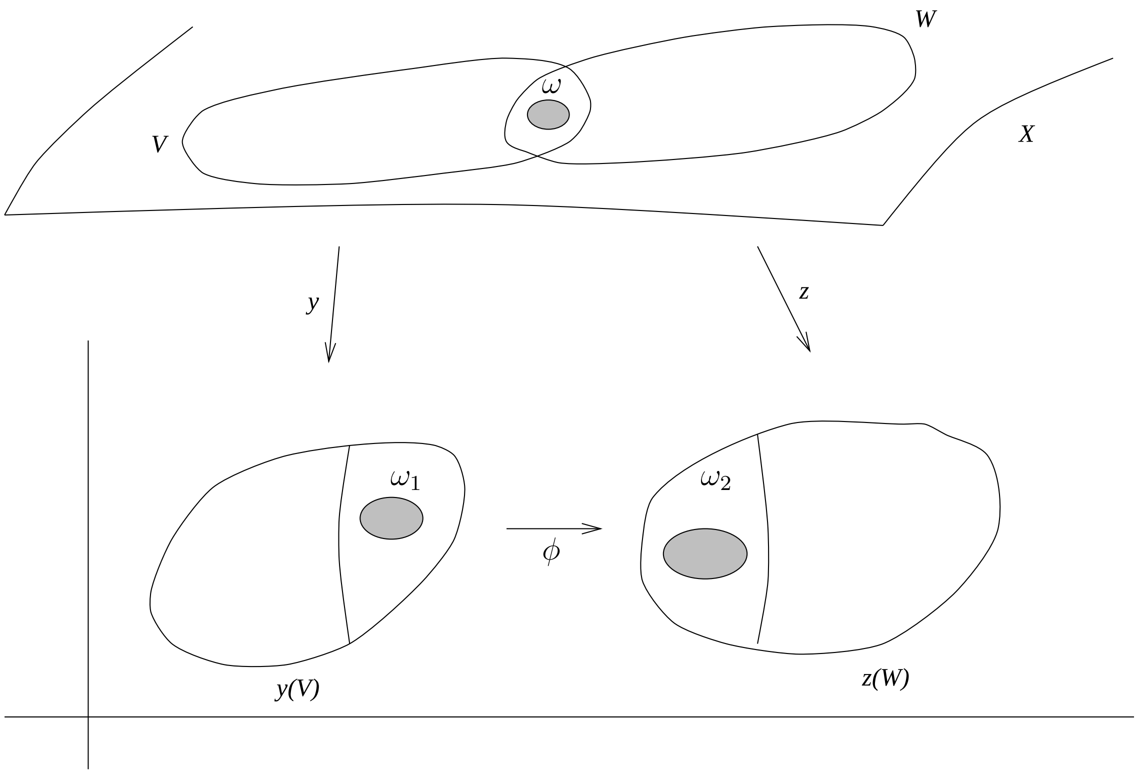

Figure 12.10

(ii) For a general n-form ω on X we write

$$\omega = \omega_1 + \cdots + \omega_r,$$

where each ω_i is an n-form satisfying the conditions of (i), and define

$$\int_X \omega = \int_X \omega_1 + \cdots + \int_X \omega_r,$$

and check that the result is independent of the choice of $\omega_1, \ldots, \omega_r$.

Definition. If X has a metric tensor then

$$\textit{volume of } X = \int_X \text{volume form.}$$

Example: If

$$\vec{v} = v^1 \frac{\partial}{\partial x} + v^2 \frac{\partial}{\partial y} + v^3 \frac{\partial}{\partial z} = \vec{\mathbf{v}}.\nabla$$

is a vector field in $\mathbb{R}^3$,

$$v = v_1\,dx + v_2\,dy + v_3\,dz = \vec{\mathbf{v}}.d\vec{\mathbf{r}}$$

is the corresponding 1-form ($v_i = v^i$), and

$$*v = v_1\,dy \wedge dz + v_2\,dz \wedge dx + v_3\,dx \wedge dy = \vec{\mathbf{v}}.\mathbf{ds}$$

is the corresponding 2-form then if u_i is a moving frame, with $u_3 = \vec{N}$ normal to surface S (see Figure 12.11), then

$$\begin{aligned}
\vec{v} &= \alpha^i u_i, \\
v &= \alpha_i u^i, \\
*v &= \alpha_1\,u^2 \wedge u^3 + \alpha_2\,u^3 \wedge u^1 + \alpha_3\,u^1 \wedge u^2,
\end{aligned}$$

where $\alpha_i = \alpha^i$. Therefore pull-back of $*v$ to X is

$$\alpha_3 u^1 \wedge u^2 = (\vec{\mathbf{v}}.\vec{\mathbf{N}})\mathbf{dS},$$

where $dS = u^1 \wedge u^2$ (area form). Therefore

$$\begin{aligned}
\int \underline{v}.\underline{dS} &= \int (\vec{v}.\vec{N})dS = \textit{flux of } \vec{v} \textit{ across } S \\
\int \vec{\mathbf{N}}.\underline{dS} &= \int dS = \text{area of } S.
\end{aligned}$$

Note. $\vec{\mathbf{v}}.\mathbf{dr}$ is work element, $\vec{\mathbf{v}}.\mathbf{ds}$ is flux element, $\vec{\mathbf{N}}.\mathbf{ds}$ is area element of vector field $\vec{\mathbf{v}}$.

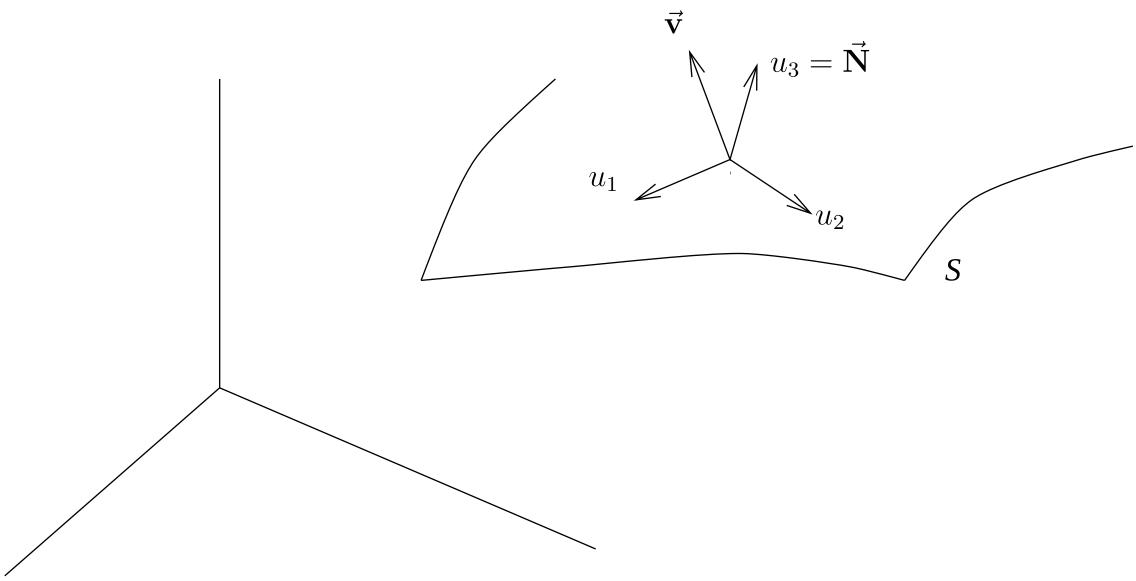

Figure 12.11

12.8 Stokes Theorem and Applications

Theorem 12.9 (Stokes). (George Gabriel Stokes 1819 - 1903, Skreen Co. Sligo) *Let ω be an $(n-1)$-form on an n-dimensional manifold X with an $(n-1)$-dimensional boundary ∂X (see Figure 12.12). Then*

$$\int_X d\omega = \int_{\partial X} i^*\omega,$$

where $\partial X \xrightarrow{i} X$ is the inclusion map.

Proof ►(Sketch) We write

$$\omega = \omega_1 + \cdots + \omega_r,$$

where each ω_i satisfies the conditions of either (i), (ii) or (iii) below, and we prove it for each ω_i. It then follows for ω.

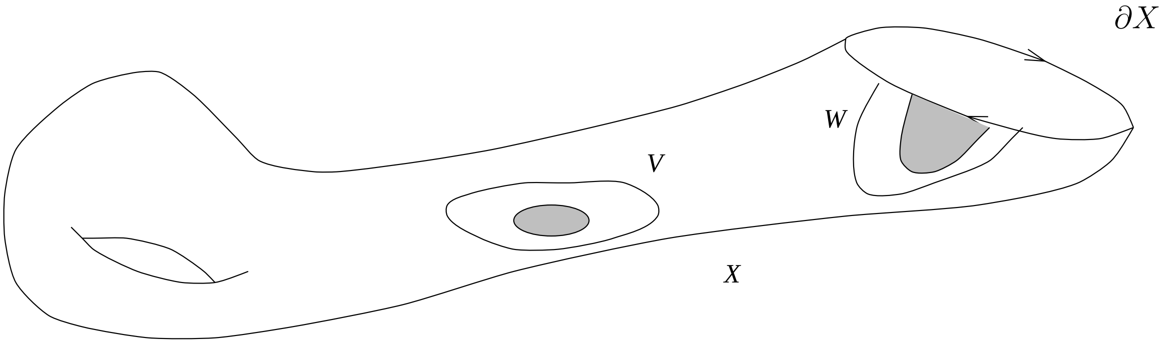

Figure 12.12

(i) Let

$$\omega = f(y^1, \ldots, y^n) dy^2 \wedge \cdots \wedge dy^n$$

on the domain V of a positively oriented coordinate system y^i such that

$$y(V) = (-1,1) \times \cdots \times (-1,1) \quad \text{(cube)},$$

ω zero outside V, and $\operatorname{supp} f$ a closed bounded subset of $y(V)$ (see Figure 12.13).

$$\int_{\partial X} i^*\omega = 0,$$

since ω is zero on ∂X.

$$d\omega = \frac{\partial f}{\partial y^1} dy^1 \wedge dy^2 \wedge \cdots \wedge dy^n.$$

Therefore

$$\begin{aligned}\int_X d\omega &= \int_{y(V)} \frac{\partial f}{\partial x^1} dx_1 dx_2 \dots dx_n \\ &= \int_{-1}^{1} \dots \int_{-1}^{1} \left[\int_{-1}^{1} \frac{\partial f}{\partial x^1} dx_1 \right] dx_2 \dots dx_n \\ &= \int_{-1}^{1} \dots \int_{-1}^{1} [f(1, x_2, \dots, x_n) - f(-1, x_2, \dots, x_n)] dx_2 \dots dx_n \\ &= 0.\end{aligned}$$

Therefore

$$\int_X d\omega = 0 = \int_{\partial X} i^* \omega.$$

For (ii) and (iii), let ω be zero outside the domain W of functions $y^1, y^2, \dots, y^n$, with y a homeomorphism:

$$y(W) = (-1, 0] \times (-1, 1) \times \dots \times (-1, 1)$$

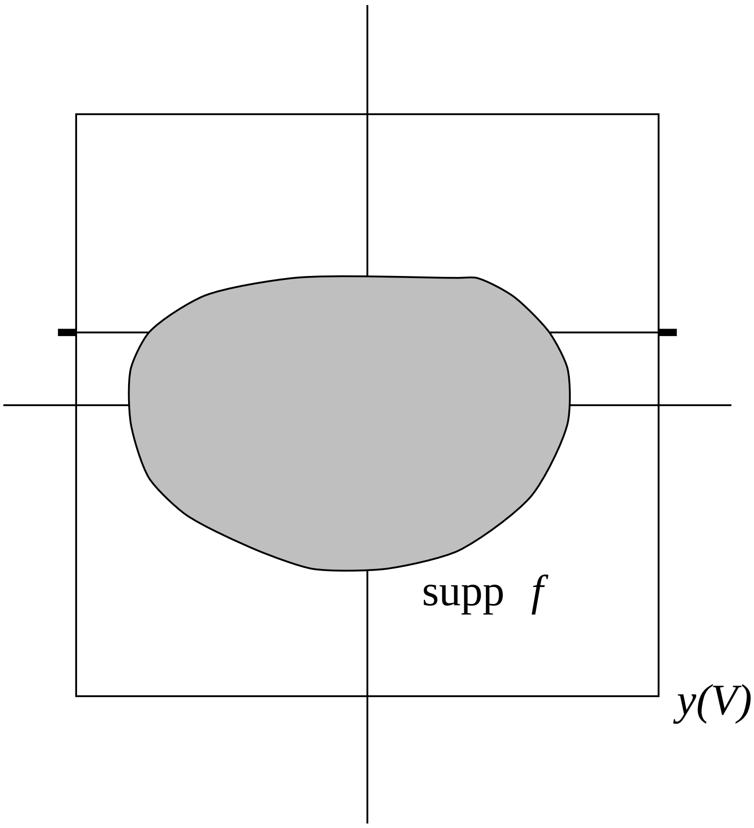

Figure 12.13

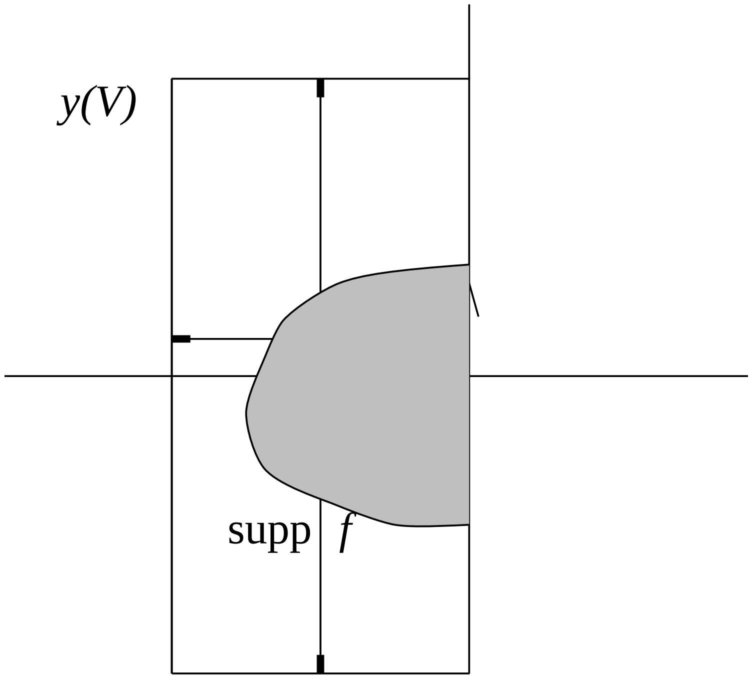

figure 12.14

(see Figure 12.14), where $y^1, y^2, \ldots, y^n$ are positively oriented coordinates on X with domain $W - (W \cap \partial X)$, $y^1 = 0$ on $W \cap \partial X$, and $y^2, \ldots, y^n$ are positively oriented coordinates on ∂X with domain $W \cap \partial X$. Then

(ii) if

$$\omega = f(y^1, y^2, \ldots, y^n) dy^2 \wedge \cdots \wedge dy^n \quad \text{(say)},$$

(supp f closed bounded subset of $y(W)$) then

$$i^*\omega = f(0, y^2, \ldots, y^n) dy^2 \wedge \cdots \wedge dy^n,$$
$$d\omega = \frac{\partial f}{\partial y^1} dy^1 \wedge dy^2 \wedge \cdots \wedge dy^n.$$

Therefore

$$\begin{aligned}\int_X d\omega &= \int_{y(W)} \frac{\partial f}{\partial x^1} dx_1 dx_2 \dots dx_n \\ &= \int_{y(W\wedge\partial X)} f(0, x_2, \dots, x_n) dx_2 \dots dx_n \\ &= \int_{\partial X} i^* \omega;\end{aligned}$$

(iii) if

$$\omega = f(y^1, \dots, y^n) dy^1 \wedge dy^3 \wedge \dots \wedge dy^n$$

(say), (supp f closed bounded subset of $y(W)$) then

$$i^* \omega = 0,$$

since $y^1 = 0$ on $W \wedge \partial X$. Also

$$d\omega = -\frac{\partial f}{\partial y^2} dy^1 \wedge dy^2 \wedge \dots \wedge dy^n.$$

Therefore

$$\int_X d\omega = -\int_{-1}^{1} \dots \int_{-1}^{0} \underbrace{\left[\int_{-1}^{1} \frac{\partial f}{\partial x^2} dx_2\right]}_{0} dx_1 \dots dx_n = 0.$$

Therefore

$$\int_X d\omega = 0 = \int_{\partial X} i^* \omega.$$

◀

Applications of Stokes Theorem:

1. In $\mathbb{R}^2$: ω 1-form (see Figure 12.17).

$$\int_{\partial D} i^*\omega = \int_\omega d\omega,$$

$$\int_{\partial D} (P\,dx + Q\,dy) = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy \quad \text{Green's Theorem.}$$

In particular:

(a)

$$\int_{\partial D} x\,dy = -\int_{\partial D} y\,dx = \frac{1}{2i}\int_{\partial D} \overline{z}\,dz = \int_D dx \wedge dy = \text{ area of } D.$$

(b) if $f = u + iv,\ \omega = f\,dz$ then

$$d\omega = \left[-\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} + i\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\right] dx \wedge dy.$$

Now

$$d\omega = 0 \Leftrightarrow f \text{ holomorphic },$$

by Cauchy-Riemann. Therefore

$$\int_{\partial D} f(z)dz = 0 \text{ if } f \text{ holomorphic} \quad \text{(Cauchy).}$$

2. In $\mathbb{R}^3$: X surface (see Figure 12.18).

$$\int_{\partial X} \underline{F}.\underline{dr} = \int_X (\nabla \times \underline{F}).\underline{dS} = \int_X (\nabla \times \underline{F}).\vec{\mathbf{N}}\,dS,$$

i.e.

$$\begin{aligned}\text{work of } F \text{ around loop } \partial X &= \text{flux of } \nabla \times F \text{ across surface } X \\ &= \text{flux of } \nabla \times F \text{ through loop } \partial X.\end{aligned}$$

3. In $\mathbb{R}^3$:

$$\int_{\partial D} \underline{F}.\vec{\mathbf{N}}\,dS = \int_{\partial D} \underline{F}.\underline{dS} = \int_D (\nabla.\underline{F})dV,$$

i.e. (see Figure 12.19)

$$\text{flux of } F \text{ out of region } D = \text{integral of } \nabla.F \text{ over interior of } D.$$

If $\nabla.\underline{F} > 0$ at a then a is a *source* for $\underline{F}$,

if $\nabla.\underline{F} < 0$ at a then a is a *sink* for $\underline{F}$,

if $\nabla.\underline{F} = 0$ then $\underline{F}$ is *source-free* (see Figure 12.20).

4. If X is n-dimensional and $\omega \wedge \eta$ an $(n-1)$-form then

$$\begin{aligned}\int_{\partial X} \omega \wedge \eta &= \int_X d(\omega \wedge \eta) \quad \text{(by Stokes)} \\ &= \int_X (d\omega) \wedge \eta + (-1)^r \int_X \omega \wedge d\eta,\end{aligned}$$

ω an r-form, by Leibsing. Therefore

$$\int_X (d\omega) \wedge \eta = \underbrace{\int_{\partial X} \omega \wedge \eta}_{\text{boundary term}} + (-1)^{r+1} \int_X \omega \wedge d\eta$$

(integration by parts) (see Figure 12.21).

Example: X connected, 3-dimensional in $\mathbb{R}^3$.

$$\begin{aligned}\int_{\partial X} f\, \nabla f.\underline{n}\, dS &= \int_{\partial X} f(\nabla f.\underline{dS}) \\ &= \int_X [\nabla f.\nabla f + f \nabla.(\nabla f)] \\ &= \int_X [\|\nabla f\|^2 + f \nabla^2 f] dV.\end{aligned}$$

Therefore

$$\nabla^2 f = 0; f \text{ or } \nabla f.\underline{n} = 0 \text{ on } \partial X \Rightarrow \nabla f = 0 \text{ on } X \Rightarrow f = 0 \text{ on } X$$

(Dirichlet Neumann).

If X has a metric tensor and no boundary then

$$\int_X (*d\omega|\eta) \operatorname{vol} = (-1)^{r+1} \int_X (*\omega|d\eta) \operatorname{vol}.$$

If (say) the metric is positive definite and n odd then $** = 1$, so putting $*\omega$ in place of ω:

$$\int_X ((-1)^r * d * \omega|\eta) \operatorname{vol} = \int_X (\omega|d\eta) \operatorname{vol}.$$

Therefore $(-1)^r * d*$ is the adjoint of the operator d. Hence $\delta = \pm * d*$ is the operator adjoint to d.

$$\Delta = d\delta + \delta d$$

is self-adjoint, and is called the *Laplacian* on f.

5. If X is the unit ball in $\mathbb{R}^n$ (see Figure 12.22), with $n \geq 2$:

$$X = \{x \in \mathbb{R}^n : \sum x_i^2 \leq 1\}$$

then ∂X is the $(n-1)$-dimensional sphere

$$\partial X = \{x \in \mathbb{R}^n : \sum x_i^2 = 1\}.$$

Theorem 12.10. *There is no C^∞ map*

$$X \xrightarrow{\varphi} \partial X$$

which leaves each point of the sphere ∂X fixed.

Proof ▶ Suppose φ exists. Then we have a commutative diagram:

$$\begin{array}{cccc} \omega & X & \xrightarrow{\varphi} & \partial X \\ & i \uparrow & \nearrow_{1} & \\ \alpha & \partial X & & \end{array}$$

where i is the inclusion map and 1 the identity map.

Let

$$\omega = x^1\, dx^2 \wedge \cdots \wedge dx^n, \quad \alpha = i^*\omega.$$

So

$$\begin{aligned} d\omega &= dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \quad \text{(volume form on } X\text{)}, \\ d\alpha &= 0, \end{aligned}$$

since $d\alpha$ is an n-form on $(n-1)$-dimensional ∂X. Therefore

$$i^*\omega = \alpha = i^*\varphi^*\alpha.$$

Therefore

$$\int_{\partial X} i^*\omega = \int_{\partial X} i^*\varphi^*\alpha.$$

Therefore volume of X is

$$\int_X d\omega = \int_X d\varphi^*\alpha = \int_X \varphi^* d\alpha \int_X \varphi^* 0 = 0.$$

This is a contradiction so the result follows. ◀

6. In $\mathbb{R}^4$, Minkowski: the electromagnetic field is a 2-form:

$$F = (\underline{E}.\underline{dr}) \wedge dt + \underline{B}.\underline{dS},$$

where $\underline{E}$ is components of electric field, and $\underline{B}$ are components of magnetic field.

One of Maxwell's equations is:

$$dF = 0$$

(the other is $d * F = J$ charge-current), i.e.

$$d[(\underline{E}.\underline{dr}) \wedge dt + \underline{B}.\underline{dS}] = 0,$$

i.e.

$$(\nabla \times \underline{E}).\underline{dS} \wedge dt + (\nabla.\underline{B})dV + \frac{\partial \underline{B}}{\partial t}.\underline{dS} \wedge dt = 0,$$

i.e.

$$\begin{aligned} \nabla \times E &= -\frac{\partial B}{\partial t}, \\ \nabla.B &= 0, \end{aligned}$$

i.e. magnetic field is source free.

Therefore electromotive force (EMF) around loop ∂X (see Figure 12.23) is

$$\begin{aligned} \int_{\partial X} \underline{E}.\underline{dr} &\overset{\text{Stokes}}{=} \int_X (\nabla \times \underline{E}).\underline{dS} \\ &\overset{\text{Maxwell}}{=} -\frac{d}{dt} \int_X \underline{B}.\underline{dS} \\ &= \text{rate of decrease of magnetic flux through loop.} \end{aligned}$$