Course 345: INTRODUCTION TO SOLITONS

Problem Set 3

Date Issued: March 31, 2008
Date due: April 7, 2008

1. Verify that the commutator of the third order differential operator

$$M = -4\frac{\partial^3}{\partial x^3} + 3u\frac{\partial}{\partial x} + 3\frac{\partial}{\partial x}u$$

and the Sturm-Liouville operator

$$L = -\frac{\partial^2}{\partial x^2} + u$$

reduces the Lax equation $L_t + [L.M] = 0$ to the KdV equation $u_t - 6uu_x + x_{xxx} = 0$.

2. Use the inverse scattering transform to find the solution of the KdV equation $u_t - 6uu_x + x_{xxx} = 0$ which satisfies $u(x, 0) = f(x)$ where

   (a) $f(x) = -\frac{9}{2}\text{sech}^2(3\text{x}/2)$ (Single soliton solution)

   (b) $f(x) = -6\text{sech}^2\text{x}$ (Two-soliton solution)

   (c) $f(x) = -12\text{sech}^2\text{x}$ (Three-soliton solution)

3. Find the asymptotic form of the three-soliton solution of the previous problem as $t \to \pm\infty$ and hence determine the phase shifts.

4. Cast the system of the ordinary differential equations

$$\frac{dx}{dt} = gy, \qquad \frac{dy}{dt} = -gx$$

where $g(x, t)$ is a given continuous function, into the equivalent form

$$L_t + [L, M] = 0,$$

finding L as some symmetric, and M as some antisymmetric, real matrix whose elements depend upon x, y and g. Show that eigenvalues of L are constant and hence deduce that $x^2 + y^2$ is constant for each solution of the system.