Course 141: MECHANICS

Problem Set 13: Solutions

* **Problem 1**

The radial energy equation of the particle is

$$E = T + V = \frac{m\dot{r}^2}{2} + \frac{J^2}{2mr^2} + \frac{k}{2r^2}$$

In the variable $u = 1/r$, we obtain

$$\frac{J^2}{2m}\left(\frac{du}{d\theta}\right)^2 + \frac{J^2}{2m}u^2 + \frac{k}{2}u^2 = E$$

and

$$J^2\left(\frac{du}{d\theta}\right)^2 + (J^2 + mk)u^2 = 2mE$$

Solution of this equation is of the form $u = \varepsilon \cos n(\theta - \theta_0)$. Indeed, in this case $\left(\frac{du}{d\theta}\right)^2 = \varepsilon^2 n^2 \sin^2 n(\theta - \theta_0)$ and identifying $J^2 n^2 = J^2 + mk$ we can see that it is a solution of the radial energy equation if

$$\varepsilon^2 = \frac{2mE}{J^2 + mk}$$

Orbits:

(a) Let $J^2 + mk = 0$. Then the radial equation reduces to

$$J^2\left(\frac{du}{d\theta}\right)^2 = 2mE$$

The solution of this simple equation is $u = \frac{\sqrt{2mE}}{J}(\theta - \theta_0) = 1/r$ and the orbit is given by $r(\theta - \theta_0) = \frac{J}{\sqrt{2mE}}$.

(b) Let $J^2 + mk \equiv -C < 0$. Then there are solutions with positive energy $E > 0$ and the radial equation reads

$$J^2\left(\frac{du}{d\theta}\right)^2 - Cu^2 = 2mE$$

The solutions are $u = a \sinh n(\theta - \theta_0)$ where $J^2 n^2 = J^2 + mk$ and $a^2 = \frac{2mE}{J^2+mk}$, so the orbit is $r \sinh n(\theta - \theta_0) = \sqrt{\frac{J^2+mk}{2mE}}$

(c) Let $J^2 + mk \equiv -C < 0$ and $E = 0$. The radial equation reads

$$J^2 \left(\frac{du}{d\theta}\right)^2 - Cu^2 = 0,$$

so, the solutions with zero total energy are $u = ae^{\pm n\theta}$ where $n^2 = C = J^2 + mk$ and the orbit is defined by the equation $re^{\pm n\theta} = 1/a =$ const.

(d) Let $J^2 + mk \equiv -C < 0$ and $E < 0$. Then the radial equation is

$$J^2 \left(\frac{du}{d\theta}\right)^2 = -2mE$$

and the solution is $u = a \cosh n(\theta - \theta_0)$. To prove in one can apply the familiar relation $\cosh^2\theta - \sinh^2\theta = 1$.

* Problem 2

(a) The potential energy of the central force $\vec{F} = F(r)\hat{r}$ is

$$V(r) = -\int_{r_0}^{r} F(r')dr' = \int_{\infty}^{r} \frac{c}{r^{3/2}} = -\frac{2c}{\sqrt{r}}$$

If $c > 0$, the radial equation reads

$$E = \frac{m\dot{r}^2}{2} + \frac{J^2}{2mr^2} - \frac{2c}{r^{1/2}}$$

and the effective potential is $U_{eff} = \dfrac{J^2}{2mr^2} - \dfrac{2c}{r^{1/2}}$.

(b) The motion is unbounded for $E \geq 0$, it is bounded between r_1 and r_2 for $E_0 < E < 0$ where E_0 is the total energy on the circular orbit of radius r_0.

(c) Because

$$U'_{eff} = -\frac{J^2}{mr_0^3} + \frac{c}{r_0^{3/2}} = 0,$$

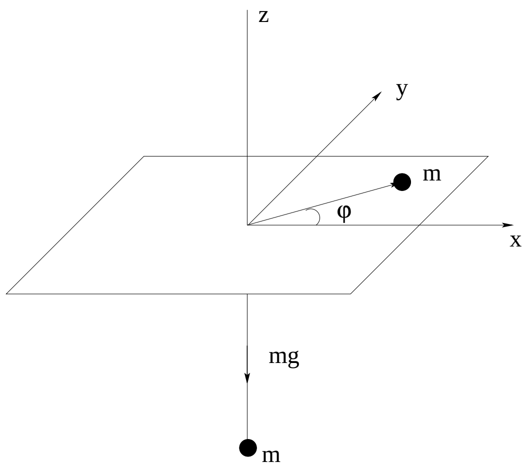
thus

$$r_0 = \left(\frac{J^2}{mc}\right)^{2/3}$$

For a circular orbit $J = mvr_0$, so the orbital period is

$$\tau = \frac{2\pi r_0}{v} = 2\pi\sqrt{\frac{m}{c}}r_0^{5/4}$$

* Problem 3



(a) The potential energy of the system is

$$V = mgz = mg(r - l)$$

The kinetic energy of the mass on the table is

$$E_1 = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2)$$

The kinetic energy of the suspended mass is

$$e_2 = \frac{m\dot{z}^2}{2} = \frac{m\dot{r}^2}{2}$$

Thus, the total energy of the system is

$$E = E_1 + E_2 + V = m\dot{r}^2 + \frac{mr^2\dot{\phi}^2}{2} + mg(r - l)$$

Conservation of the angular momentum $\vec{J} = mr^2\dot{\phi}\hat{k}$ gives

$$E = m\dot{r}^2 + \frac{J^2}{2mr^2} + mg(r - l) = \text{ const}$$

The effective radial equation then can be obtained by differentiation of this formula w.r.t. time:

$$\frac{dE}{dt} = 0 = \left(2m\ddot{r} - \frac{J^2}{mr^3} + mg\right)\dot{r}$$

or, simple $2m\ddot{r} - \frac{J^2}{mr^3} + mg = 0$. This equation admits a solution for a circular orbit with $\ddot{r} = 0, r = r_0 = $ const:

$$mg = \frac{J^2}{mr_0^3},$$

thus $r_0 = \left(\dfrac{J^2}{m^2 g}\right)^{1/3}$. Because of the conservation of the angular momentum $\dot{\phi} = \frac{J}{mr_0^2} = \sqrt{\frac{g}{r_0}} =$ const.

(b) Because $E = m\dot{r}^2 + U_{eff}$ where $U_{eff} = \frac{J^2}{2mr^2} + mg(r-l)$ the physical motion is restricted by the condition $U_{eff} \leq E$. For $E \geq \frac{J^2}{2mr^2}$ the motion is unbounded, i.e., both masses end up on the table. For $U_{min} \leq E \leq \dfrac{J^2}{2mr^2}$ the motion is bounded. The minimum U_{min} corresponds to the stationary motion with $r = r_0 =$ const.

(c) Expansion of the effective potential U_{eff} in Teylor series in the vicinity of the equilibrium distance r_0 gives:

$$U_{eff}(r_0 + \delta) \approx U_{eff}(r_0) + \left.\frac{\partial U_{eff}}{\partial r}\right|_{r=r_0} \delta(t) + \frac{1}{2}\left.\frac{\partial^2 U_{eff}}{\partial r^2}\right|_{r=r_0} \delta^2(t)$$

The equation of motion yields $\left.\dfrac{\partial U_{eff}}{\partial r}\right|_{r=r_0} = 0$ and the frequency of small oscillations is defined as $\omega^2 = k/(2m)$ where

$$k = \left.\frac{\partial^2 U_{eff}}{\partial r^2}\right|_{r=r_0} = \frac{3J^2}{mr_0^4} = \frac{3mg}{r_0}$$

* **Problem 4**

The energy of the planet before and after explosion explosion is $E_1 = T_1 + V_1$ and $E_2 = T_2 + V_2$, respectively. The kinetic energy conserves, i.e., $T_1 = T_2$ while the potential energy decreases as $V_2 = V_1/2$ because $V(r) = -\dfrac{GMm}{r}$ and $M \to M/2$. On the other hand, for a circular orbit $E_1 = \left|\dfrac{V_1}{2}\right|$, thus $T_1 = -\dfrac{V_1}{2} = T_2$ and after explosion

$$E_2 = -\frac{V_1}{2} + \frac{V_1}{2} = 0$$

That means the motion becomes parabolic.

* **Problem 5**

The corresponding equation of motion is

$$m\ddot{\vec{r}} + k\vec{r} = 0$$

The solution of this isotropic oscillator problem is ($\omega^2 = k/m$)

$$\vec{r} = \vec{A}\cos\omega t + \vec{B}\sin\omega t$$

where $\vec{A}, \vec{B}$ are arbitrary constant vectors which are determined by the initial conditions. Thus, the motion is periodic with the period $\tau = 2\pi/\omega$ which is independent of the initial conditions.

For any fixed angle θ we can rotate the vectors $\vec{A}, \vec{B}$ as

$$\vec{A} = \vec{a}\cos\theta - \vec{b}\sin\theta; \qquad \vec{B} = \vec{a}\sin\theta + \vec{b}\cos\theta$$

or, equivalently

$$\vec{a} = \vec{A}\cos\theta + \vec{B}\sin\theta; \qquad \vec{b} = -\vec{A}\sin\theta + \vec{B}\cos\theta$$

The choice of the angle θ is given by the condition that $(\vec{a}\cdot\vec{b}) = 0$, i.e., $\tan 2\theta = 2(\vec{A}\cdot\vec{B})/(A^2 - B^2)$. Then the equation of motion becomes

$$x = a\cos(\omega t - \theta); \qquad y = b\sin(\omega t - \theta); \qquad z = 0$$

This gives the equation of the orbit

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which defines an ellipse with centre at the origin, and semi-axes a, b.