Bergman-Szegő kernel asymptotics in weakly pseudoconvex finite type cases

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Riemann mapping theorem

Let $\mathbb{D}^n := \{z \in \mathbb{C}^n | \, |z| < 1\} \subset \mathbb{C}^n$.

Theorem (Riemann mapping theorem 1851)

Let $U \subset \mathbb{C}$ open, proper, simply connected. Then $U \cong \mathbb{D}^1$ (biholomorphic)

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Theorem (Poincare 1907)

In $\mathbb{C}^2$, the polydisk and disk $\mathbb{D}^1 \times \mathbb{D}^1 \ncong \mathbb{D}^2$ (not biholomorphic).

Poincare's proof computes:
Aut $(\mathbb{D}^n) = PSU(n,1) :=$
$\left\{ T = \begin{bmatrix} A & b \\ c & d \end{bmatrix} | T \begin{bmatrix} I_n & \\ & -1 \end{bmatrix} T^* = \begin{bmatrix} I_n & \\ & -1 \end{bmatrix}, \det T = 1 \right\} / S^1$ where $T.z = \frac{Az+b}{cz+d}$

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Questions:

-find generalization of RMT in higher dimensions?

-biholomorphism classification of domains in higher dimensions
-more robust biholomorphism invariants?

Bergman kernel

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$L^2(U) := \left\{ f : U \to \mathbb{C} \text{ measurable} \mid \|f\|^2_{L^2(U)} = \int_U |f|^2 \, d\mathrm{vol} < \infty \right\}$
$H^0_{(2)}(U) := \left\{ f \in L^2(U) \,|\, \bar{\partial} f = 0 \right\} \subset L^2(U)$

Bergman projection: $\Pi_U : L^2(U) \to H^0_{(2)}(U) \subset L^2(U)$
Bergman kernel: $\Pi_U \in L^2(U \times U)$ satisfies

$$(\Pi_U f)(z) = \int_U \Pi_U(z, z') f(z') \, d\mathrm{vol}(z')$$

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Properties:

1. $\Pi_U(z, z') = \sum_{j=1}^{\infty} \varphi_j(z) \overline{\varphi_j(z')}$, for $\{\varphi_j\}_{j=1}^{\infty}$ ONB of $H^0_{(2)}(U)$.
2. $\overline{\Pi_U(z, z')} = \Pi_U(z', z)$
3. $\Pi_U(z, z')$ hol./antihol. in z/z', smooth in the interior
4. (U bounded) $\Pi_U(z, z) > 0$ in the interior

Bergman metric

The Bergman metric on U is defined via

$$g_{ij}^{U} = \partial_{z_i}\partial_{\overline{z_i}}\left[\ln \Pi_U\left(z,z\right)\right]$$

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Theorem (Bergman 1931)

Let $F : U_1 \to U_2$ be a biholomorphism between two domains. Then it is an isometry $F_* g^{U_1} = g^{U_2}$.

Proof.

From defining property $\Pi_{U_1}\left(z, w\right) = \det \left(\frac{\partial F}{\partial z} \right) \left[\Pi_{U_2}\left(F\left(z\right), F\left(w\right) \right) \right] \overline{\det \left(\frac{\partial F}{\partial w} \right)}$.
Mixed partials of Jacobians are zero. $\square$

Computations of Π_U

1. $U = \mathbb{D}^n$ (disk)

ONB: $\sqrt{\frac{\binom{\alpha+n}{n}}{\pi^n}} z^\alpha$

$$\Pi_{\mathbb{D}^n}\left(z, z'\right) = \frac{n!}{\pi^n} \frac{1}{\left(1 - z\overline{z'}\right)^{n+1}}$$

$g^{\mathbb{D}^n}$ corresponds to Poincare disk model for hyperbolic space

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2. (D'Angelo '78)

$U = E_p \coloneqq \left\{|z_1|^2 + |z_2|^{2p} \leq 1\right\}$ (ellipsoid)

$$\Pi_{E_p}(z, z') = \frac{2}{\pi^2} \frac{1}{p} \frac{\left(1 - z_1\overline{z_1'}\right)^{\frac{2}{p}-2}}{\left[\left(1 - z_1\overline{z_1'}\right)^{\frac{1}{p}} - z_2\overline{z_2'}\right]^3} + \frac{2}{\pi^2} \frac{p-1}{p} \frac{\left(1 - z_1\overline{z_1'}\right)^{\frac{2}{p}-2}}{\left[\left(1 - z_1\overline{z_1'}\right)^{\frac{1}{p}} - z_2\overline{z_2'}\right]^2}$$

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Computing curvatures: $\mathbb{D}^1 \times \mathbb{D}^1 \not\cong \mathbb{D}^2$, $E_p \not\cong E_{p'}$ for $p \neq p'$ (not biholomorphic).

Fefferman's theorem

Another strategy for $\mathbb{D}^1 \times \mathbb{D}^1 \ncong \mathbb{D}^2$: look for behaviour of biholomorphisms near boundary (boundary of polydisk is non-smooth)

Conjecture: (cf. Krantz '13) Let $U_1, U_2 \subset \mathbb{C}^n$ smoothly bounded. Then any biholomorphism $F : U_1 \to U_2$ extends smoothly to the boundary.

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Theorem (Fefferman '74)

Let $U_1, U_2 \subset \mathbb{C}^n$ smoothly bounded and strongly pseudoconvex. Then $U_1 \cong U_2$ (biholomorphic) $\iff$ $\exists$CR diffeomorphism $\partial U_1 \cong \partial U_2$.

Proof.

Uses Bergman kernel expansion. Given $U = \{\rho < 0\}$, $d\rho|_{\partial U} \neq 0$ (boundary defining function) , then

$$\begin{aligned}\Pi_U(z,z) &= a(z)\,\rho^{-n-1} + b(z)\ln(-\rho)\\ &\sim \sum_{j=0}^{\infty} a_j(x)\,\rho^{-n-1+j} + \sum_{j=0}^{\infty} b_j(x)\,\rho^j \ln(-\rho), \quad \text{as } \rho \to 0,\end{aligned}$$

for $a(z), b(z) \in C^\infty\left(\overline{U}\right)$, $z = (x,\rho)$ local coord near boundary. Study geodesic flow for Bergman metric near boundary to obtain boundary extension. □

CR manifolds

A CR manifold $\left(X^{2n+1}, T^{1,0}X \subset T_{\mathbb{C}}X\right)$
-codim: $\dim_{\mathbb{C}} T^{1,0}X = n$
-non-degeneracy: $T^{1,0}X \cap \overline{T^{1,0}X} = \emptyset$
-integrability: $\left[T^{1,0}X, T^{1,0}X\right] \subset T^{1,0}X$

$f : X_1 \to X_2$ is a CR map iff $f_* T^{1,0}X_1 \subset T^{1,0}X_2$

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Examples:
1. $X = \partial U$, $U \subset \mathbb{C}^n$smoothly bounded domain.
Where $T^{1,0}X = T^{1,0}\mathbb{C}^n \cap T_{\mathbb{C}}X$.

2. Y cpx manifold. $\left(L, h^L\right)$ Hermitian holomorphic.
CR manifold $X = S^1 L$, HX =Chern connection, $T^{1,0}X = \ker\left(J - i\right) \subset HX \otimes \mathbb{C}$

Pseudoconvexity, Finite type

CR manifold $\left(X^{2n+1}, T^{1,0}X \subset T_{\mathbb{C}}X\right)$

Levi distribution: $HX = \mathrm{Re}\ \left[T^{1,0}X \oplus T^{0,1}X\right]$
Levi form: $\mathscr{L} \in (HX^*)^{\otimes 2} \otimes (T_xX/H_xX)$
$$\mathscr{L}(U,V) := [[U,V]] \in T_xX/H_xX$$

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$x \in X$ is weakly/strongly pseudoconvex $\iff \mathscr{L}_x$ is pos. def./semi-def.
$x \in X$ is finite type $\iff HX$ is bracket generating at x

type of a point $x \in X$:
$r(x) = 1 + \min \#$ brackets in HX necessary to generate TX

type of strongly pseudoconvex point $x \in X$ is $r(x) = 2$.

Tangential CR complex & Szegő kernel

Tangential CR operator: $\bar{\partial}_b^q : C^\infty\left(\Lambda^q T^{0,1*}X\right) \to C^\infty\left(\Lambda^{q+1}T^{0,1*}X\right)$,
Defined by Leibniz rule following: $\left(\bar{\partial}_b^0 f\right)\left(\bar{Z}\right) = \bar{Z}\left(f\right)$, $f \in C^\infty\left(X\right)$, $\overline{Z} \in T^{0,1}X$.

$\bar{\partial}_b \circ \bar{\partial}_b = 0$ (uses integrability)
Kohn-Rossi cohomology: $H^{0,q}\left(X\right) := \ker\left(\bar{\partial}_b^q\right)/\mathrm{Im}\left(\bar{\partial}_b^{q-1}\right)$

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Choosing Hermitian metric on $T^{1,0}X$ & volume form μ.
Kohn Laplacian: $\Box_b^q := \bar{\partial}_b^{q-1}\left(\bar{\partial}_b^{q-1}\right)^* + \left(\bar{\partial}_b^q\right)^* \bar{\partial}_b^q$

Szegő projector: $\Pi_b^q : L^2\left(\Lambda^q T^{0,1*} X\right) \to \ker\left(\bar{\partial}_b^q\right)$

Hodge theory

Kohn described Hodge theory when Levi form non-degenerate.

Theorem (Kohn '65)

Let $\left(X^{2n+1}, T^{1,0}X\right)$ *strongly pseudoconvex CR manifold. Then for each* $q \neq 0, n$ *the Kohn Laplacian* $\Box_b^q$ *is hypoelliptic and with a corresponding Hodge theorem* $H^{0,q}(X) = ker \left(\Box_b^q\right)$.

The $\ker \left(\Box_b^q\right)$ can be infinite dimensional for $q = 0, n$.

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The $\ker\ \left(\square_b^q\right)$ can be infinite dimensional for $q = 0, n$.

Theorem (Kohn '65)

Let $\left(X^{2n+1}, T^{1,0}X\right)$ strongly pseudoconvex. Furthermore assuming $\bar{\partial}_b = \bar{\partial}_b^0$ has closed range, $\exists G : H^s\left(T^{0,1*}X\right) \to H^{s+\frac{1}{2}}\left(X\right)$ such that $\Pi_b^0 = I - G\bar{\partial}_b$

Szegő parametrix

Interesting to describe the singularities of its Schwartz kernel.

Theorem (Boutet de Monvel-Sjöstrand '75)

Let $\left(X^{2n+1}, T^{1,0}X\right)$ *be strongly pseudoconvex CR manifold. Assume* ∂_b *has closed range. Then near each* $x \in X$ *there exist coordinates* $\left(\underbrace{x_1 \ldots, x_{2n}}_{=x'}, x_{2n+1}\right)$ *such that*

$$\begin{aligned}\Pi_b^0(x,y) &= \int_0^\infty dt e^{it\Psi(x,y)} a(t;x,y) \\ &= \int_0^\infty dt e^{it(x_{2n+1}-y_{2n+1})} \underbrace{e^{it\Phi(x',y')} a(t;x,y)}_{=b(t;x,y)}\end{aligned}$$

Phase: $\overline{\Phi(x',y')} = \Phi(y',x')$, $Im\Phi(x',y') \geq C|x'-y'|^2$, $x' = y' \iff \Phi = 0$.
Amplitude: $a(t;x,y) \in S^n_{t,cl}$, $a \sim \sum_{j=0}^\infty t^{n-j} a_j(x,y)$

Note: regrouped amplitude $b(t;x,y) \in S^n_{t,\frac{1}{2},\mathrm{cl}}$
(lies in a more general class; $\partial_x^\alpha \partial_y^\beta \partial_t^\gamma b = O\left(t^{n+\frac{1}{2}(|\alpha|+|\beta|)-\gamma}\right)$)

Bergman asymptotics

Specialize to $X = \partial U$.
Consider Poisson operator

$$P : C^{\infty}(X) \to C^{-\infty}(U)$$
$$\square_U P = 0, \gamma P = I.$$

Theorem (Boutet de Monvel '71)

The Poisson operator maps $P : H^s(X) \to H^{s+\frac{1}{2}}(U)$. *Furthermore* $P^*P : C^{\infty}(X) \to C^{\infty}(X)$ *is elliptic, injective pseudodifferential of order* -1 *with* $\sigma(P^*P)^{-1} = \sigma(\Delta_X)$.

Furthermore P approximately relates the Bergman-Szegő projectors

$$\Pi_U " = " P (P^*P)^{-1} \Pi_b^0 P$$

(at highest order)

Using the above recover/refine Fefferman's theorem

Greens function, pointwise bounds

Consider $\left(X^3, T^{1,0}X\right)$ weakly pseudoconvex, finite type.
Let maximum number of brackets be $r := \max_{x \in X} r\left(x\right)$.

Theorem (Christ '89)

Let $\left(X^3, T^{1,0}X\right)$ weakly pseudoconvex, finite type. Assume $\bar{\partial}_b$ has closed range. There exists (microlocal) $G : H^s\left(X\right) \to H^{s+\frac{1}{r}}\left(X\right)$ such that $\Pi_b^0 = I - G\bar{\partial}_b$.

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Theorem (Christ '89)

Let$(X^3, T^{1,0}X)$ weakly pseudoconvex, finite type. Assume $\bar{\partial}_b$ has closed range. Near any point $x' \in X$ of type $r(x')$ there exists coordinates (x_1, x_2, x_3) centered at x'such that

$$|\partial_x^\alpha \Pi(x,0)| \le C_\alpha \left[d^H(x,0)\right]^{-2-r-\alpha_1-\alpha_2-r\alpha_3}$$

$$d^H(x,0) = |x_1| + |x_2| + |x_3|^{1/r(x')} .$$

($d^H(x,0)$ is equivalent to the sub-Riemannian CC distance between $x, 0$).

Similar bounds for boundaries of weakly pseudoconvex finite type domains in $\mathbb{C}^2$: McNeal '89, Nagel-Rosay-Stein-Wainger '89

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Theorem (Hsiao-S. '20)

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$$\Pi\left(x, 0\right) = \int_0^\infty dt\, e^{itx_3} b\left(x; t\right) + C^\infty\left(X\right)$$

where $b \sim t^{\frac{2}{r}}\left[\sum_{j=0}^\infty t^{-\frac{j}{r}} b_j\left(t^{\frac{1}{r}}x_1, t^{\frac{1}{r}}x_2\right)\right] \in S^{\frac{2}{r}}_{\frac{1}{r},cl}\left(\mathbb{R}^2_{x_1,x_2} \times \mathbb{R}_t\right)$, $b_j \in \mathcal{S}\left(\mathbb{R}^2\right)$.

Christ estimates are equivalent to $b \in S^{\frac{2}{r}}_{\frac{1}{r}}\left(\mathbb{R}^2_{x_1,x_2} \times \mathbb{R}_t\right)$
(i.e. $\partial_t^k \partial_x^\alpha b = O\left(t^{m-k+\frac{|\alpha|}{r}}\right)$ without classical expansion).

Bergman asymptotics

Specializations:

Theorem (Hsiao-S. '20)

Let $X^3 = \partial U$ *be boundary of weakly pseudoconvex, finite type domain* $U = \{\rho < 0\} \subset \mathbb{C}^2$. *For any point* $x' \in X = \partial U$ *on the boundary, of type* $r = r(x')$, *the Bergman kernel satisfies the asymptotics*

$$\Pi_U(z,z) \sim \sum_{j=0}^{\infty} \frac{1}{(-\rho)^{2+\frac{2}{r}-\frac{1}{r}j}} a_j + \sum_{j=0}^{\infty} b_j (-\rho)^{j} \log(-\rho),$$

as $z \to x'$ *for some set of reals* a_j, b_j *with* $a_0 > 0$.

Fefferman '74 (strongly pseudoconvex case), D'Angelo '78 (ellipsoids), Boas-Straube-Yu '95 (h-extendible/semiregular domains), Kamimoto '98, '04 (tube domains, toric domains)...

Bergman asymptotics

Theorem (Marinescu-S. '18)

Let Y^2 Riemann surface. (L, h^L) Hermitian, holomorphic, semi-positive. Assume that R^L vanishes to finite order at each point. The Bergman kernel admits on-diagonal expansion

$$\Pi_k(y,y) \sim k^{\frac{2}{r}}\left[b_0(y) + b_1(y)k^{-\frac{2}{r}} + b_2(y)k^{-\frac{4}{r}}\ldots\right]$$

where $r = r(y) = 2 + ord_y(R^L)$.

The above proved by local index theory method as in Bismut-Lebeau '91, Dai-Liu-Ma '06, Ma-Marinescu '07.

Higher dimensions?

Theorem (Ohsawa '84)

Let $U \subset \mathbb{C}^n$ *weakly pseudoconvex and* $\nu(x') = dim\ ker \mathscr{L}_{x'}$. *Then*

$$\Pi_U(z,z) \gtrsim (-\rho)^{-n-1+\nu(x')}$$

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Theorem (McNeal '92)

$U \subset \mathbb{C}^n$ *weakly pseudoconvex & D'Angelo finite type. Then*

$$\underbrace{\|v\|_{g^U}}_{Bergman\ metric\ norm} \gtrsim |v|\,(-\rho)^{-\delta-\varepsilon}$$

Where δ *=sub-elliptic regularity gain in* $\bar{\partial}-$*Neumann problem.*

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Where δ =sub-elliptic regularity gain in $\bar{\partial}-$Neumann problem.

Theorem (Diederich-Ohsawa '95)

$U \subset \mathbb{C}^n$ weakly pseudoconvex. Then

$$Berg.\ dist(x,x') \geq -1 + C_1 \ln|\ln C_2(-\rho(x))|.$$

Proof Sketch

Step 1. Let

$$\begin{aligned} Z &= a_j(x)\,\partial_{x_j} && \text{CR vector field} \\ \tilde{Z} &= \tilde{a_j}(z)\,\partial_{z_j} + \tilde{b_j}(z)\,\partial_{\bar{z}_j} && \text{almost analytic extension} \end{aligned}$$

Find *almost analytic BRT coordinates* $(w_1(z), w_2(z), w_3(z))$ on $\mathbb{C}^3$ such that

$$\tilde{Z} = \frac{1}{2}(\partial_{w_1} + i\partial_{w_2}) - \frac{i}{2}(\partial_{w_1}\varphi + i\partial_{w_2}\varphi)\,\partial_{w_3} \quad \text{for}$$

$$\varphi = \underbrace{\varphi_0(w_1, w_2)}_{\text{homogeneous, real coefficients}} + O\left(|w|^{r+1}\right).$$

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$$\varphi = \underbrace{\varphi_0(w_1, w_2)}_{\text{homogeneous, real coefficients}} + O\left(|w|^{r+1}\right).$$

Step 2.
The local Bergman kernel $B_t(p, p')$ for the Hermitian metric $e^{-t\varphi(p_1,p_2)}dp$, $p = \mathrm{Re}\, w$ satisfies symbolic estimates/expansion in $t \to \infty$ by local index method.

Proof sketch

Step 3.

Construct almost analytic extensions for Szegő kernel $\Pi(x, x')$ on $X = \{\mathrm{Re} z = 0\}$ and t-Fourier transform of $B_t(p, p')$ on $\mathbb{R}^3_p = \{\mathrm{Re} w = 0\}$.
This part requires Christ estimates.

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Step 4.

Relate almost analytic extensions Π and t-Fourier transform of B_t.

Relation is pointwise Boutet de Monvel-Sjöstrand theorem.
This requires constructing a calculus of symbols $S^{m,k}_{\frac{1}{r},\mathrm{cl}}$ satisfying Christ estimates.

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Step 5.

For Fefferman's expansion; the relation between Bergman & Szegő kernels Π_U, Π via the Poisson operator still holds.
Also uses requires solution of $\bar{\partial}$-Neumann problem for weakly pseudoconvex domains $U \subset \mathbb{C}^2$ (Kohn '72).

Thank you.